Symmetric extensions of polytopes

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Abstract

This writing are the notes of the talk given by myself in the seminar "Extension of polytopes" organized by Prof. Dr. Raman Sanyal and Dr. Arnau Padrol at Freie Universität Berlin on the 6th of May of 2015. The main objective is to show the Yannakakis' method to prove lower bound for symmetric extensions of polytopes. In particular, it is shown how this method work in the case of regular polygons where one can appreciate how the requirement of symmetry matters for extensions of polytopes.

1 Notations and preliminaries

Before starting, let's recall some definition and fix some notations regarding discrete geometry and group theory.

Regarding discrete geometry, an **affine map of polyhedra** $f : P \to Q$ will be an affine map $f : \mathrm{aff}(P) \to \mathrm{aff}(Q)$ such that $f(P) \subseteq Q$. Following this abuse of language, we will call this affine map of polyhedra an **affine projection** if $f(P) = Q$ and **affine isomorphism** if it has an affine inverse (or, equivalently, that is bijective). Also, recall that a **slack variable of** P is an affine map $f : \mathrm{aff}(P) \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in P$ and that it is said to be **normalized at** $p \in \mathrm{relint}(P)$ if $f(p) = 1$ and **irreducible** if $f^{-1}(0) \cap P$ is a facet of P. Finally, $\mathrm{bc}(P)$ will denote the **barycenter** of P, $V(P)$ the **set of vertices** of P and $F_p(P)$ the **set of irreducible slack variables of** P **normalized at** $p \in \mathrm{relint}(P)$.

Regarding group theory, $G \curvearrowright X$ will denote an action of G on X using the notation gx for left actions and x^g for right actions. Here, $\mathrm{Orb}_G(x)$ will be the **orbit** of $x \in X$,$\mathrm{Stab}_G(x)$ the **stabilizer** of $x \in X$ and $\ker(G \curvearrowright X)$ the **kernel** of $G \curvearrowright X$ (which is the subgroup of G that act trivially on X). Also, we will say that $G \curvearrowright X$ is **transitive** if there is a unique orbit, **free** all stabilizers are trivial and **faithfull** if it has trivial kernel. Finally, Σ_X and A_X will denote, respectively, the **symmetric group** and an **alternating group** of a set X and we will use σ indistinctly for the permutation $\sigma \in \Sigma_n$ and the linear map $(x_1, \ldots, x_n) \mapsto \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$ of $\mathbb{R}^n$.

2 Symmetries and polytopes/polyhedra

As any kind of object in mathematics subjected to some structure, it is natural to consider the symmetries that preserves the structure of polytopes. However, our usual "schizophrenia" in discrete geometry makes us to distinguish between the symmetries that preserve the geometric structure (formally, affine structure) and the ones that preserve the combinatorial structure (formally, face lattice) of the polytope.

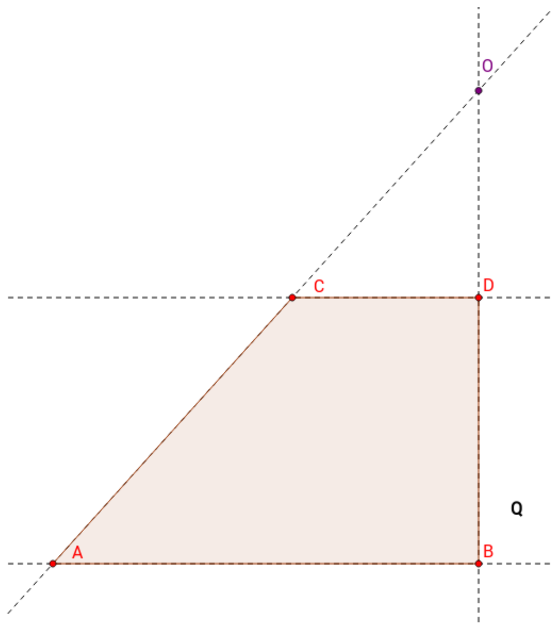
Definition 2.1. *Let Q be a polyhedron, then*

*g) a **geometric automorphism** of Q is an affine map $f : Q \to Q$ such that $f(Q) = Q$,*

*c) a **combinatorial automorphism** of Q is a poset-automorphism of $\mathscr{L}(Q)$.*

The group of geometric automorphism of Q will be denoted by $\mathrm{Aut}_g(Q)$ and the group of combinatorial automorphism by $\mathrm{Aut}_c(Q)$.

The following polygon (drawn with [IGI15])



have not the same combinatorial and affine automorphisms due to that affine intersection O not detected by the combinatorics. Therefore there is in general a difference between considering geometric and combinatorial automorphisms.

We recall/state without proof the following obvious facts:

- $\mathrm{Aut}_g(P)$ acts faithfully on $\mathscr{L}(P)$ via $gF = g(F)$..
- The above action restrict to a faithful action on $V(P)$.

- For any affine isomorphism of polytopes $f : P \to Q$, $f(\mathrm{bc}(P))\mathrm{bc}(Q)$. In particular, every geometric automorphism fix the barycenter.

- $\mathrm{Aut}_g(P)$ acts faithfully on the set of fundamental non-negative maps of P centered at the barycenter $F_{\mathrm{bc}(P)}(P)$ via $f^g = f \circ g$.

3 Symmetric extensions of polytopes

Following the fundamental ideas of Yannakakis [Yan91] and their deepening by Kaibel, Pashkovich and Theis [KPT12], we will define and develop the concepts of symmetric extensions, sectional system (the first tool to prove lower bounds), the Yannakakis' action (the second tool) and the Yannakakis' method to prove lower bounds.

An extension of a polytope is a polyhedron that projects onto it, but this extension may not preserve the symmetries of this polytope. In order to take into account the lifting of symmetries, the concept of symmetric extension is necessary.

Definition 3.1. *Let P be a polytope and G a subgroup of* $\mathrm{Aut}_g(P)$*, a **G-symmetric extension** of P is an affine projection of polyhedra*

$$\pi : Q \to P$$

such that for all $g \in G$*, there is some* $\chi_g \in \mathrm{Aut}_g(P)$ *such that*

$$\pi \circ \chi_g = g \circ \pi.$$

*In this context, the **size** of* $\pi : Q \to P$*,* $\mathrm{size}(\pi : Q \to P)$*, will be the number of facets of Q, a **lift** of* $g \in G$ *will be an element* $\chi \in \mathrm{Aut}_g(Q)$ *satisfying* $\pi \circ \chi = g \circ \pi$ *and we will say that* $\pi : Q \to P$ *is **bounded** if Q is a polytope.*

*Here, the **G-symmetric extension complexity** of P is the minimum size among all G-symmetric extensions of P, i.e., the number given given by*

$$\mathrm{xc}_G(P) := \min\{\mathrm{size}(\pi : Q \to P) \mid \pi : Q \to P \text{ is a } G\text{-symmetric extension of } P\}.$$

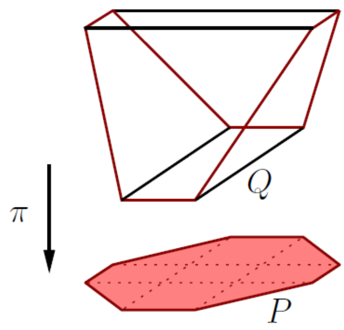
Note that when G is the trivial subgroup, this reduces to the usual definition of extensions of polytopes. Also, note that the lifting of elements of G is element by element so it is not necessarily well-behaved; in particular, this means that in general we do not have a group homomorphism from G into $\mathrm{Aut}_g(Q)$, but from a subgroup of $\mathrm{Aut}_g(Q)$ onto G[1].

[1] This is radically different from definition 2.10 of [GPT13] which *a priori* will enclosed only a subclass of the cases that permits our definition. So we must ask: are these two definitions essentially different?

Recall that the question[2] that we will answer here for regular polygons and permutahedra –in this second case up to constant– is:

Let P be a polytope and G a subgroup of $\mathrm{Aut}_g(P)$, what is $\mathrm{xc}_G(P)$?

Here, note that the known projection for the standard simplex $\Delta \to P$ give G-symmetric extensions of P for every subgroup G of $\mathrm{Aut}_g(P)$ and that a extension is symmetric or not depending on the group we are working with as shown by the following extension of a regular octogon P ("stolen" from [FRT12, Figure 1])



which is H-symmetric for H is the subgroup generated by the orthogonal reflections along the horizontal and vertical axis, but not with respect its full geometric automorphism group.

3.1 Sectional system of a section

As extensions are no more than projections for polytopes, it is natural to introduce the concept of section for these. As the usual definition of section as a right inverse would be too restrictive, we define sections of vertices.

Definition 3.2. *Let P be a polytope and $\pi : Q \to P$ an extension, a* ***section*** *of $\pi : Q \to P$ is a map*

$$s : V(P) \to Q$$

such that $\pi \circ s = \mathrm{id}_{V(P)}$.

Using Farkas lemma II and III [Zie07, p. 41-42], one proves the following proposition which will be the first tool for for proving inexistence results of certain size extensions by *reductio ad absurdum*.

[2] A question in the reverse direction which could be interesting is the following one: can there be arbitrarily large asymmetric extension of a polytope? Or more concretely: given a polytope P and a subgroup G of $\mathrm{Aut}_g(P)$, are there arbitrarily large extensions of P that are not G-symmetric?

Proposition 3.1 (Sectional system). *Let P be a polytope, $\pi : Q \to P$ an extension and $q \in \mathrm{relint}(Q)$, then for all sections $s : V(P) \to Q$ of $\pi : Q \to P$ and slack variables of P $g : \mathit{aff}(P) \to \mathbb{R}$, the system*

$$\begin{cases} \sum_{v\in V(P)} \lambda_v (f \circ s)(v) \geq 0 & \big(f \in F_q(Q)\big) \qquad (\mathrm{S}\geq) \\ \sum_{v\in V(P)} \lambda_v g(v) < 0 & \qquad (\mathrm{S}<) \end{cases}$$

has no solution $\lambda \in \mathbb{R}^{V(P)}$ such that $\sum_{v\in V(P)} \lambda_v \geq 0$.

For now on, we will refer to the above system of inequalities as the **sectional system** of a section.

3.2 Yannakakis' action of a symmetric section

Before starting, we note that (by using quotients of affine spaces to quotient out the recession cone) without loss of generality we can assume that every symmetric extension is bounded. So this hypothesis does not limit the results in this subsection.

Lemma 3.2 (Bounding lemma). *Let P be a polytope and G a subgroup of $\mathrm{Aut}_g(P)$. If P has a a G-symmetric extension of size t, then P has a bounded G-symmetric extension of size $\leq t$.*

In the first place, let's introduce the adequate concept of section for a symmetric extension: symmetric sections.

Definition 3.3. *Let P be a polytope, G a subgroup of $\mathrm{Aut}_g(P)$ and $\pi : Q \to P$ a G-symmetric extension, a* ***symmetric section*** *of $\pi : Q \to P$ is a section $s : V(P) \to Q$ such that for all $g \in G$ and every lift of it $\chi_g \in \mathrm{Aut}_g(Q)$,*

$$\chi_g \circ s = s \circ g.$$

It is not clear that symmetric sections must exist (and they don't if we have a symmetric extension $\pi : Q \to P$ with Q not pointed). However, after our initial restrictions, they do.

Lemma 3.3. *Let P be a polytope, G a subgroup of $\mathrm{Aut}_g(P)$ and $\pi : Q \to P$ a bounded G-symmetric extension of P, then there exist a symmetric section $s : V(P) \to Q$.*

Proof. Take

$$\begin{aligned} s : V(P) &\to Q \\ v &\to \mathrm{bc}(\pi^{-1}(v)) \end{aligned}$$

Then the invariance of the barycenter with respect affine isomorphism together with the action of G on the set of fibers of $\pi : Q \to P$, $\{\pi^{-1}(v) \cap Q \,|\, v \in V(P)\}$, given by $g\pi^{-1}(v) = \pi^{-1}(gv)$. □

In the second and last place, we will construct what we will call the **Yannakakis' action** of a symmetric section.

Theorem 3.4 (Yannakakis' action). *Let P be a polytope, G a subgroup of $\mathrm{Aut}_g(P)$, $\pi : Q \to P$ a bounded G-symmetric extension of P and $s : V(P) \to Q$ a symmetric section of it, then G acts on the set*

$$\{f \circ s \,|\, f \in F_{\mathrm{bc}(Q)}(Q)\}$$

via $(f \circ s)^g = (f \circ s) \circ g$.

Proof. The only not so trivial part is that this is a well-defined action, but, due to the fact that s is a symmetric section, for every $g \in G$ and lift $\chi_g \in \mathrm{Aut}_g(Q)$,

$$(f \circ s)^g = (f \circ s) \circ g = (f \circ \chi_g) \circ s$$

is an element in the considered set since $f \circ \chi_g \in F_{\mathrm{bc}(Q)}(Q)$. □

It is important to note that all the statements presented here can be generalized easily to a more general setting of "symmetric extensions of objects", i.e., the ideas used here are group theoretical in essence.

3.3 Yannakakis' method for lower bounds

Once our two tools have been developed, it is time of developing the Yannakakis' method for establishing lower bounds for the symmetric extension complexity of polytopes. Suppose that we have a polytope P with G a (non-trivial) subgroup of $\mathrm{Aut}_g(P)$ and that we want to prove that

$$\mathrm{xc}_G(P) \geq t.$$

In order to show this, we argue by *reductio ad absurdum*. If this were false, by the bounding lemma (3.2), there would exist a bounded G-symmetric extension $\pi : Q \to P$ of size $< t$. Here, we consider the symmetric section

$$s : V(P) \to Q,$$

guaranteed to exist by lemma 3.3, and the sectional system associated to it (given by inequalities (S$\geq$) and (S$<$)). Here, using the Yannakakis' action of G we can gather the terms of this sectional system. More concretely, fix $f \in F_{\mathrm{bc}(Q)}(Q)$ and suppose that the orbit decomposition of $V(P)$ with respect G is given by

$$V(P) = \mathrm{Orb}_G(v_1) \cup \cdots \cup \mathrm{Orb}_G(v_\alpha)$$

and that the orbit decomposition of each orbit $\mathrm{Orb}_G(v_i)$ with respect $\mathrm{Stab}_G(f \circ s)$ is given by

$$\mathrm{Orb}_G(v_i) = \mathrm{Orb}_{\mathrm{Stab}_G(f\circ s)}(v_{i,1}^f) \cup \cdots \cup \mathrm{Orb}_{\mathrm{Stab}_G(f\circ s)}(v_{i,\beta_{i,f}}^f),$$

then we can rewrite the inequality (S≥) as

$$\sum_{i=1}^{\alpha} \left(\sum_{j=1}^{\beta_{i,f}} (f \circ s)(v_{i,j}^f) \left(\sum_{w \in \mathrm{Orb}_{\mathrm{Stab}_G(f\circ s)}(v_{i,j}^f)} \lambda_w \right) \right) \geq 0.$$

This gathering of terms, it becomes easier to argue since we have reduce the number of unknown coefficients $(f \circ s)(v_{i,j}^f)$ that we cannot control in the inequality.

In brief and as conclusion of the previous digression, the **Yannakakis' method for lower bounds** consists in using the Yannakakis' action to gather the terms in the inequalities (S≥) in order to reduce the unknown coefficients that we cannot control using the information about how G acts on $V(P)$ and how the $\mathrm{Stab}_G(f \circ s)$ look like[3].

4 Lower bounds for the symmetric extension complexity of regular polygons

In this section, we will apply the Yannakakis' method in order to obtain lower bounds of regular polygons. Furthermore, we will contrast this results in lower bounds for symmetric extensions with the extension complexity of these polytopes in order to show that symmetry requirements can really affect the size of it. In other words, paraphrasing Kaibel, Pashkovich and Theis in [KPT12], symmetry matters!

Up to translation, rotation and dilation, every **regular polygon** of n vertices is of the form

$$P_2(n) := \mathrm{conv}\left\{ \left(\cos\left(\frac{2\pi}{n}k\right), \sin\left(\frac{2\pi}{n}k\right) \right) \mid 0 \leq k < n \right\}.$$

It is a common fact that

$$\mathrm{Aut}_g(P_2(n)) \cong D_{2\cdot n}$$

[3]As we will see with the permutahedron (and in general in any case with a complicated group G), this question can be hard because *a priori* the only thing that we will know is that these subgroups have index bounded by the size of the hypothetical extension.

where $D_{2\cdot n}$ is the **dihedral group of order** $2\cdot n$ formed by the n rotations and the n orthogonal reflections with respect axis of symmetries. Here, C_n will denote the cyclic normal subgroups of rotations and R the generator given by the rotation by $\frac{2\pi}{n}$ radians in the positive sense. It is important to note that C_n acts in a free and transitive way on the set of vertices of $P_2(n)$, so every vertex can be written uniquely as $R^k e_1$ with $0 \leq k < n$.

If symmetry restrictions are dropped from the extension of a regular polygon, we can see using reflections [FRT12] that

$$\mathrm{xc}(P_2(n)) = \Theta(\log n).$$

In contrast to this minimality result for extensions of regular polygons, we have that in the symmetric case the situation is as bad (or good) as it can be as shown by the following theorem[4].

Theorem 4.1 (Symmetric extension complexity of regular polygons)**.** *For every n,*

$$\mathrm{xc}_{C_n}(P_2(n)) = n.$$

Proof. Suppose that the statement is false and let $\pi : Q \to P_2(n)$ be a bounded C_n-symmetric extension of size $< n$ and $s : V(P) \to Q$ its symmetric extension. Then for every $f \in F_{\mathrm{bc}(Q)}(Q)$,

$$\mathrm{Stab}_{C_n}(f \circ s) = C_{d_f} := \langle R^{d_f} \rangle$$

for some $d_f > 1$ divisor of n (just apply the structure theorem of cyclic groups together with the fact that the orbit of f has $|C_n : \mathrm{Stab}_{C_n}(f \circ s)| < n$ elements) and so, via Yannakakis's method, (S≥) is rewritten as

$$\sum_{a=0}^{d_f - 1} \left(\sum_{\substack{x \equiv a \pmod{d_f} \\ 0 \leq x < n}} \lambda_{R^x e_1} \right) (f \circ s)^{R^a}(e_1) \geq 0$$

for all proper divisor d of n and non-negative integer $a < d$.

Here, considering the slack variable for $P_2(n)$

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto 1 - x \end{aligned}$$

and taking

$$\lambda_{R^x e_1} = 1 - g(R^x e_1) = \cos\left(\frac{2\pi}{n} x\right)$$

[4] And only original result in this text!

gives a solution to the system of inequalities of the sectional system thanks to the fact that

$$\sum_{x=0}^{N-1} \cos\left(\alpha + \frac{2\pi}{N}x\right) = 0$$

for all $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$ as a consequence of being the barycenter of any regular polygon centered at the origin 0. Hence, by *reductio ad absurdum*, $P_2(n)$ has no C_n-symmetric extension of size $< n$ and so $\mathrm{xc}_{C_n}(P_2(n)) = n$ as desired. □

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