Semialgebraic sets

Semialgebraic sets are the class of geometric objects that can be described by real polynomials and inequalities. A way of describing a semialgebraic set is to use formulas. Formulas are expression obtained by combining atoms of the form

- $\varphi(x) < 0$, $\varphi(x) \geq 0$,
- $\varphi(x) > 0$ and $\varphi(x) \leq 0$,
- $\varphi(x) = 0$, $\varphi(x) \neq 0$,

which represent the most basic semialgebraic sets; using

- negations ($\neg$), which represent complements;
- conjunctions ($\land$), which represent intersections; and
- disjunctions ($\lor$), which represent unions.

Formulas should be seen as “recipes” telling us how to construct the described set from the most basic ones.

**Example** Consider the formula $(x^2 + y^2 - 1 \leq 0) \land (x + y = 0)$.

In this formula, we have three atoms: $x^2 + y^2 - 1 \leq 0$, which represents the filled unit circle; $(x + y = 0)$ or $(x + y > 0)$ or $(x + y < 0)$, a line through the origin; and $(3y - x^2 < 0)$, the points below a parabola.

These can be seen below:

- The red curve represents $x^2 + y^2 - 1 \leq 0$.
- The blue curve represents $(x + y = 0)$.
- The green curve represents $3y - x^2 < 0$.

Following the formula, on the left side, $(x^2 + y^2 - 1 \leq 0) \lor (x + y = 0)$, the points are not in $(y - x^2 < 0)$ to take the points not in $(y - x^2 < 0)$. These operations give the sets below:

- $(x^2 + y^2 - 1 \leq 0) \lor (3y - x^2 < 0)$
- $(x^2 + y^2 - 1 \leq 0) \land (3y - x^2 < 0)$

In the last step, $(x^2 + y^2 - 1 \leq 0) \lor (3y - x^2 < 0)$ tells us to take only those points coming at the same time both from $(x^2 + y^2 - 1 \leq 0)$ or $(3y - x^2 < 0)$.

Our current result

Theorem. There is a numerical algorithm, numerically stable, with input polynomial $q$-tuples $f$ and lax formulas $\Phi$ (i.e. without $\equiv$, $\equiv$, $\equiv$) using the polynomial in $f$ that computes the homology groups of the semialgebraic set described by $\Phi$ in weak exponential time for $f$ uniformly distributed on the sphere.

This result is good, because we expect such an algorithm (numeric or not) to take exponential time at least.

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