On this day...

the mathematian

Michel Chasles

would have

his 229th birthday

Famous for Chasles' identity:

$$\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac}$$

A whole career reduced to this!

Symbolic Computation Seminar
NCSU

Computing Numerically the Homology of a Semialgebraic Set

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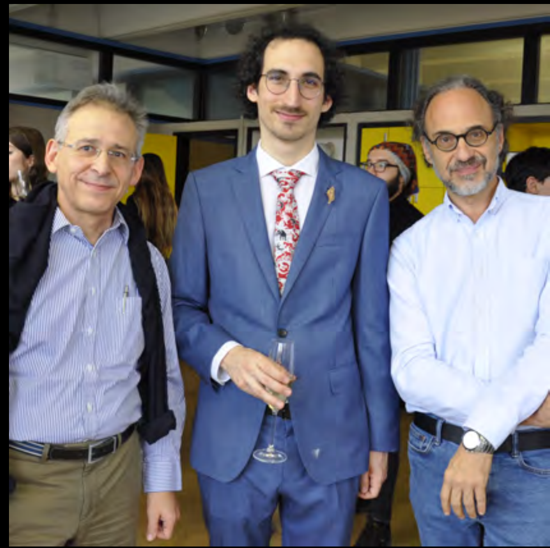

References.

Computing the Homology of Semialgebraic Sets. I: Lax Formulas
Foundations of Computational Mathematics, 20(1):71-119, 2020.
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Computing the Homology of Semialgebraic Sets. II: General Formulas
Foundations of Computational Mathematics, 21(5):1279–1316, 2021.
DOI 10.1007/s10208-020-09483-8

Condition and Homology in Semialgebraic Geometry.
PhD thesis, Technsiche Universität Berlin, DepositOnce Repository, December 2019.
DOI 10.14279/depositonce-9453

Why do we care?

Central problem
in computational
semi alg. geometry

(with applications)

THE THEOREM:

There is an algorithm Hom that given a q-tuple $p \in \prod_{i=1}^{q} \mathbb{R}[X_1, \dots, X_n]_{\le d_i}$ of polynomials and a semialgebraic formula Φ over p of size s, it computes the homology groups of

$$S_{\mathrm{aff}}(p, \Phi) \subseteq \mathbb{R}^n$$

semialg. set defined by (p, Φ)

in time

$$\mathcal{O}\left(qs\left(128\, n\, D\, \overline{\kappa}_{\mathrm{aff}}(p)\right)^{10n(n+2)}\right)$$

THE THEOREM: (Homogeneous Version)

There is an algorithm Hom that given a q-tuple $\mathfrak{f} \in \prod_{i=1}^{q} \mathbb{R}[X_0, \dots, X_n]_{d_i}$ of homogeneous polynomials and a semialgebraic formula Φ over $\mathfrak{f}$ of size s, it computes the homology groups of

$$S(\mathfrak{f}, \Phi) \subseteq \mathbb{S}^n$$

spherical semialg. set def. by $(\mathfrak{f}, \Phi)$

in time

$$\mathcal{O}\left(qs\left(128\, n\, D\, \overline{\kappa}(\mathfrak{f})\right)^{10n(n+2)}\right)$$

THE THEOREM: (Homogeneous Version +)

There is an algorithm Hom_ℓ that given a q-tuple $\mathfrak{g} \in \prod_{i=1}^{q} \mathbb{R}[X_0, \ldots, X_n]_{d_i}$ of homogeneous polynomials and a semialgebraic formula Φ over $\mathfrak{g}$ of size s, it computes the first ℓ homology groups of

$$S(\mathfrak{g}, \Phi) \subseteq \mathbb{S}^n$$

spherical semialg. set def. by $(\mathfrak{g}, \Phi)$

in time

$$\mathcal{O}\left(q s \left(128\, n\, D\, \overline{\kappa}(\mathfrak{g})\right)^{10 n(\ell+2)}\right)$$

THE PROB. THEOREM: (Homogeneous Ver)

Given a random KSS q-tuple f,

i.e. $f_i = \sum \binom{d_i}{\alpha}^{1/2} c_{i,\alpha} X^{\alpha}$ with the

the $c_{i,\alpha} \sim N(0,1)$ i.i.d., then HOM_{ℓ}

runs with

$$\mathcal{O}\left(s\,(qD)^{\mathcal{O}(n^2 \ell)} \, \Omega^{10n(\ell+1)} \right)$$

with prob $\geq 1 - 1/\Omega$

Also smoothed analysis and more robust probabilistic models

Single exponential in n

with high probability!

Improves state of the art!

Algorithm at a glance

0. Condition number estimation
1. Reduction to Lax Case
2. Reduction to Basic Cases
3. Basic case

Condition Number

$$\bar{\kappa}(f,x) := \max_{\substack{I \subseteq [q] \\ \# I \leq n+1}} \frac{\|f_I\|_W}{\sqrt{\|f_I(x)\|^2 + \sigma_{|I|}\left(\Delta_I^{-1/2} D_x f_I\right)}}$$

where $f_I := (f_i)_{i \in I}$ $\quad D_x f_I : T_x \mathbb{S}^n \to \mathbb{R}^I$

tangent map

$\Delta = \operatorname{diag}(d_i)_{i \in I}$

$\sigma_{|I|}$ $|I|$-sing. val.

$$\bar{\kappa}(f) := \max_{x \in \mathbb{S}^n} \bar{\kappa}(f,x) \in [1,\infty)$$

Interpretation of $\bar{\kappa}(\delta)$

$$\bar{\kappa}(\delta) < \infty$$

$$\Leftrightarrow: \forall i, \ \mathcal{Z}_S(\delta_i) \text{ smooth hypersur.}$$

$$\& \ \forall I \subseteq [q],$$

$$\bigcap_{i \in I} \mathcal{Z}_S(\delta_i) \text{ transversal}$$

Note: If $|I| \geq n+1$, then $\bigcap_{i\in I} \mathcal{Z}_S(\delta_i)$ Transversal is empty

Estimation of $\bar{\kappa}(g)$

Prop. $\mathbb{S}^n \ni x \longmapsto 1/\bar{\kappa}(g,x)$ is D-Lipschitz

Cor. If $\mathcal{G} \subseteq \mathbb{S}^n$ satisfies

$$d_H(\mathcal{G}, \mathbb{S}^n) < \varepsilon,$$

& $\max\{\bar{\kappa}(g,x) \mid x \in \mathcal{G}\}\, D\, \varepsilon < 1/2$, then

$$\bar{\kappa}(g) \leq 2 \max\{\bar{\kappa}(g,x) \mid x \in \mathcal{G}\}$$

Reduction to Lax Case I

Gabrielov-Vorobjov Construnction:

$$\Gamma B_{\delta,\varepsilon}(g,\Phi) = S(g,\tilde{\Phi})$$

where in $\tilde{\Phi}$
$$\begin{cases} g_i = 0 \rightsquigarrow |g_i| \leq \varepsilon \|g_i\|_w \\ g_i > 0 \rightsquigarrow g_i \geq \delta \|g_i\|_w \\ g_i < 0 \rightsquigarrow g_i \leq -\delta \|g_i\|_w \end{cases}$$

$$\Gamma B_{\underline{\delta},\underline{\varepsilon}}(g,\Phi) = \bigcup_i \Gamma B_{\delta_i,\varepsilon_i}(g,\Phi)$$

Gabrielov-Vorobjov theorem:

If $0 << \varepsilon_1 << \delta_1 << \varepsilon_2 << \delta_2 << \dots << \delta_m << 1$,

then $\exists\, H_k(\Gamma B_{\underline{\delta},\underline{\varepsilon}}(g,\Phi)) \to H_k(S(g,\Phi))$: iso for $k \leq m-1$

& surj. for $k = m$

Reduction to Lax Case II

Gabrielov-Vorobjov Construnction:

$$\Gamma B_{\delta,\varepsilon}(\gamma, \Phi) = S(\gamma, \tilde{\Phi})$$

where in $\tilde{\Phi}$
$$\begin{cases} \gamma_i = 0 \rightsquigarrow |\gamma_i| \leq \varepsilon \|\gamma_i\|_w \\ \gamma_i > 0 \rightsquigarrow \gamma_i \geq \delta \|\gamma_i\|_w \\ \gamma_i < 0 \rightsquigarrow \gamma_i \leq -\delta \|\gamma_i\|_w \end{cases}$$

$$\Gamma B_{\underline{\delta},\underline{\varepsilon}}(\gamma, \Phi) = \bigcup_i \Gamma B_{\delta_i,\varepsilon_i}(\gamma, \Phi)$$

Quantitative Gabrielov-Vorobjov theorem:

If $0 < \varepsilon_1 < \delta_1 < \varepsilon_2 < \dots < \delta_m < 1/\sqrt{2}\,\bar{R}(\gamma)$

then $\exists\, H_k(\Gamma B_{\underline{\delta},\underline{\varepsilon}}(\gamma, \Phi)) \to H_k(S(\gamma, \Phi))$: iso for $k \leq m-1$

& surj. for $k = m$

Reduction to basic case I

Main Idea:

- $\mathcal{X}_i^{\alpha}$ sample of $S(\delta_i\, \alpha\, 0)$

- If for all basis $\phi = \bigwedge_{i \in I} (\delta_i\, \alpha_i\, 0)$

  $\bigcap \mathcal{X}_i^{\alpha_i}$ good sample of $S(\delta, \phi)$,

  then for all semialg. formula Φ,

  $\Phi(\mathrm{Simp}(\mathcal{X}_i^{\alpha})) \simeq S(\delta, \Phi)$

Reduction to basic case II

Ingredients:

Explicit Functional Nerve Theorem

$\pi : \check{C}_\varepsilon(\mathcal{X}) \longrightarrow B_\varepsilon(\mathcal{X})$ homology-equiv.

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$\sum t_i [x_i] \longmapsto \sum t_i x_i$

Homological Inclusion-Exclusion Transfer

$$\left.\begin{array}{l} \gamma : X \to Y \text{ cont.} \\ X = \bigcup X_i,\ Y = \bigcup Y_i \\ \gamma(X_i) \subseteq Y_i + \text{Tech.} \end{array}\right\}$$

$\forall I,\ \gamma : \bigcap_{i \in I} X_i \to \bigcap_{i \in I} Y_i$ homology equiv.

$\Longrightarrow \gamma : X \to Y$ hom. equiv.

Also hom. iso up to k and surj for $(k+1)$-homology

Basic Case TDA ingredients

Niyogi-Smale-Weinberger Thm

$$\exists\, d_H(x,X) < \varepsilon < \tfrac{1}{2}\tau(X) \Rightarrow B_\varepsilon(x) \overset{\sim}{\hookleftarrow} X$$

Hausdorff distance

hom. equiv

reach

$$\tau(X) := \inf\{r>0 \mid \exists\, p\in\mathbb{R}^n, x,\tilde{x}\in X \;\; d(p,X)=d(p,x)=d(p,\tilde{x})=r \;\&\; x\neq\tilde{x}\}$$

Attali-Lieutier-Salinas Thm

$$\exists\, d_H(x,X) < \varepsilon < \frac{1}{5}\tau(X) \Rightarrow VR_\varepsilon(x) \overset{\sim}{\hookleftarrow} \check{C}_\varepsilon(x)$$

Vietoris-Rips

Čech

Basic Case Approx Results

Reach Bound

$$\tau\Big(Y \cap \bigcap X_i\Big) \geq \min_{I} \tau\Big(Y \cap \bigcap_{i\in I} \partial X_i\Big)$$

Reduction to reach of boundaries

ϕ basic semialg. form $\Rightarrow \exists\, D^{3/2}\,\overline{\kappa}(\mathfrak{f})\,\tau\big(S(\mathfrak{f},\phi)\big) > 1$

Condition-based bound of reach

Sampling Thm

$\mathcal{G} \subseteq \mathbb{S}^n$, $d_H(\mathcal{G},\mathbb{S}^n) < r$

If $\sqrt{2}\,D^{1/2}\,\overline{\kappa}(\mathfrak{f})\,r < 1$, then for all lax formulas

Φ, $d_H\big(\mathcal{G}\cap S_{D^{1/2}r}(\mathfrak{f},\Phi),\, S(\mathfrak{f},\Phi)\big) < \sqrt{2}\,D^{1/2}\,\overline{\kappa}(\mathfrak{f})\,r$

$\leftarrow$ Relaxation: $\mathfrak{f} \geq 0 \rightsquigarrow \mathfrak{f} \geq -D^{1/2}r, \ldots$

Improvements

- Using $\|\delta\|_\infty := \max_i \max_{x \in \mathbb{S}^n} |\delta_i(x)|$ instead of $\|\delta\|_W$ reduces in one n the exponent of prob. run-time
- Can we implement it?

Eskerrik asko

bere arretagatik!