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On the Number of Real Zeros of Random Sparse Polynomial Systems

or...

how few real zeros does a random fewnomial system have?

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Joint work with...

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Both pictures taken in San Antonio, Texas, USA

How many zeros does it have?

$$\begin{cases} a_1 + b_1 X + \gamma_1 Y + c_1 XYZ^d = 0 \\ a_2 + b_2 X + \gamma_2 Y + c_2 XYZ^d = 0 \\ a_3 + b_3 X + \gamma_3 Y + c_3 XYZ^d = 0 \end{cases}$$

generically... d in $\mathbb{C}^3$ (intersection theory)

≤ 2 in $\mathbb{R}^3$

≤ 1 in $\mathbb{R}^3_+$

Kushnirenko's Question:

$A_1, \dots, A_n \subseteq \mathbb{N}^n \qquad t_i = \# A_i$

$$f := \begin{cases} f_1 := \sum_{\alpha \in A_1} f_{1,\alpha} X^\alpha \\ \quad \dots \\ f_n := \sum_{\alpha \in A_n} f_{n,\alpha} X^\alpha \end{cases}$$

What do we call such a system with few terms?

fewnomial

$$Z_r(f, \mathbb{R}^n_+) := \{\xi \in \mathbb{R}^n_+ \mid f(\xi) = 0, \det D_\xi f \neq 0\}$$

Is there a bound of the form

$$\# Z_r(f, \mathbb{R}^n_+) \leq U(t_1, \dots, t_n, n)?$$

Kushnirenko's Question:

Propagandistic version

Do fewnomial systems have few positive zeros?

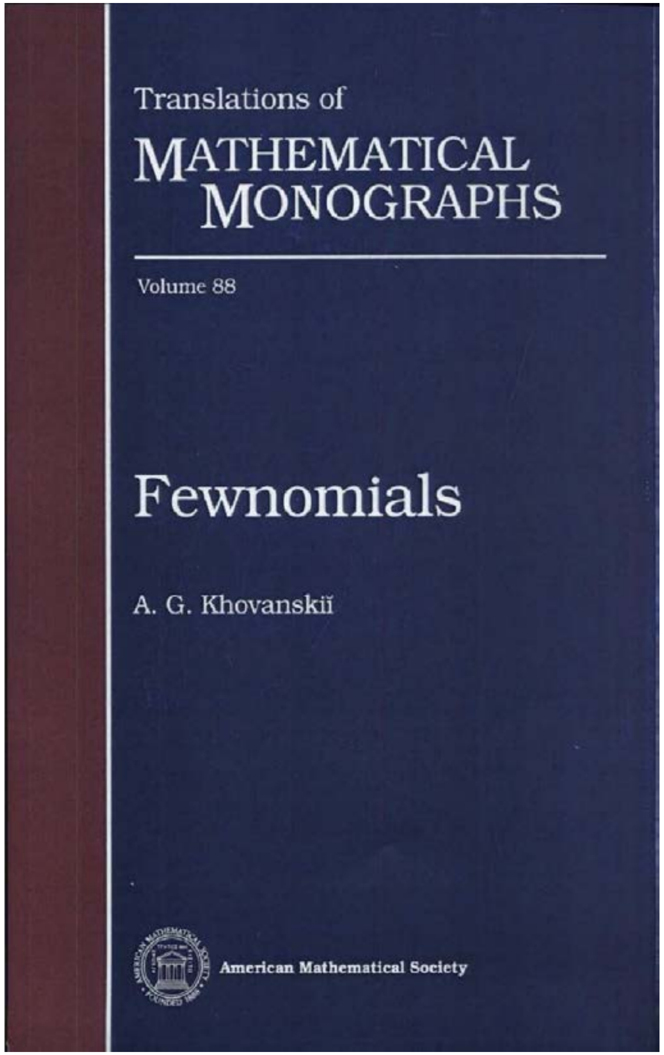
polynomial → МНОГОЧЛЕН → МАЛОЧЛЕН → fewnomial

$\underbrace{\text{many}\quad\text{nomials}}$ $\underbrace{\text{few}\quad\text{nomials}}$

poconomio (es)

gutxinomio (eu)

Khovanskii's Answer & Refinements



$A \subseteq \mathbb{N}^n \qquad t = \#A \ (\le t_1 + \cdots + t_n)$

$\mathcal{F} := \{ f_i := \sum_{\alpha \in A} f_{i,\alpha} X^\alpha \quad (i = 1, \ldots, n)$

(Khovanskii, 1991)

$$\#Z_r(\mathcal{F}, \mathbb{R}^n_+) \le 2^{\binom{t-1}{2}} (n+1)^{t-1}$$

(Bihan & Sottile; 2007)

$$\#Z_r(\mathcal{F}, \mathbb{R}^n_+) \le \frac{e^2+3}{4} 2^{\binom{t-n-1}{2}} n^{t-n-1}$$

Kushnirenko's Question:

A more precise version

$A_1, \ldots, A_n \subseteq \mathbb{N}^n \qquad t_i = \# A_i$

⚠ STILL OPEN FOR $n=2$!!!

$$g := \begin{cases} g_1 := \sum_{\alpha \in A_1} g_{1,\alpha} X^\alpha \\ \quad \cdots \\ g_n := \sum_{\alpha \in A_n} g_{n,\alpha} X^\alpha \end{cases}$$

$$Z_r(g, \mathbb{R}^n_+) := \{ \zeta \in \mathbb{R}^n_+ \mid g(\zeta) = 0, \det D_\zeta g \neq 0 \}$$

Is there a bound of the form

$$\# Z_r(g, \mathbb{R}^n_+) \leq \mathrm{poly}(t_1, \ldots, t_n)^n \ ?$$

Can we say

SOMETHING

for a random fewnomial system?

spoiler: yes!

Reasons to care...

about random fewnomial systems

Better understanding of typical behaviour

— not the worst case, but the ~~average~~ typical case

Robust probabilistic results might be a pathway to deterministic ones

$$\sup_{g} \# Z_r(g, \mathbb{R}^n_+) = \sup_{e \geq 1} \left(\mathbb{E}_F \# Z_r(F, \mathbb{R}^n_+)^e \right)^{1/e}$$

Our Random Model

$A_1, \ldots, A_n \subseteq \mathbb{R}^n$ w/ $t_i = \# A_i$ fixed

$$f := \begin{cases} f_1 := \sum_{\alpha \in A_1} f_{1,\alpha} X^\alpha \\ \cdots \\ f_n := \sum_{\alpha \in A_n} f_{n,\alpha} X^\alpha \end{cases}$$

random fewnomial system

such that the $f_{i,\alpha}$ are independent,

centered ($\mathbb{E} f_{i,\alpha} = 0$)

& Gaussian

presented at MEGA2019!

(Bürgisser, Ergür & TC; 2019) & (Bürgisser, ISSAC'23)

are particular cases of our model

Our Result: The Easy Form

$A_1, \ldots, A_n \subseteq \mathbb{R}^n$ w/ $t_i = \# A_i$ fixed

$F := \{ f_i := \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha \quad (i = 1, \ldots, n) \}$

s.t. the $f_{i,\alpha}$ are independent, centered & Gaussian

$$\mathbb{E}_F \, Z_r(F, \mathbb{R}_+^n) \leq \frac{1}{4^n} \prod_{i=1}^{n} t_i (t_i - 1)$$

Our Result: Unit Variance Case

$A_1, \dots, A_n \subseteq \mathbb{R}^n$ w/ $t_i = \# A_i$ fixed

$F := \{ F_i := \sum_{\alpha \in A_i} F_{i,\alpha} X^\alpha \quad (i = 1, \dots, n) \}$

s.t. the $F_{i,\alpha}$ are independent, centered & Gaussian

IF (VM1) for all i and $\alpha \in A_i$, $\mathbb{V}(F_{i,\alpha}) \leq 1$

(VM2) For all i and $\alpha \in A_i$ vertex of $P_i := \mathrm{conv}(A_i)$,

$$\mathbb{V}(F_{i,\alpha}) = 1$$

THEN

$$\mathbb{E}_F \, Z_r(F, \mathbb{R}_+^n) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n P_i\right) \prod_{i=1}^n (t_i - 1)$$

#vertices

Minkowski sum

Improves (Bürgisser, ISSAC'23)

Our Result: The Unmixed Case

$A \subseteq \mathbb{R}^n$ w/ $t = \# A$ fixed

$F := \{ F_i := \sum_{\alpha \in A} f_{i,\alpha} X^\alpha \quad (i = 1, \ldots, n) \}$

s.t. the $f_{i,\alpha}$ are independent, centered & Gaussian

IF $\begin{cases} \text{(VM1) for all } i \text{ and } \alpha \in A,\ \mathbb{V}(f_{i,\alpha}) \leq 1 \\ \text{(VM2) for all } i \text{ and } \alpha \in A \text{ vertex of } P := \operatorname{conv}(A),\ \mathbb{V}(f_{i,\alpha}) = 1 \end{cases}$

OR

$\mathbb{V}(f_{i,\alpha})$ only depends on $\alpha \in A$

THEN

$$\mathbb{E}_F Z_r(F, \mathbb{R}^n_+) \leq \frac{n+1}{4^n} \binom{t}{n+1}$$

Improves (Bürgisser, Ergür & TC: 2019)

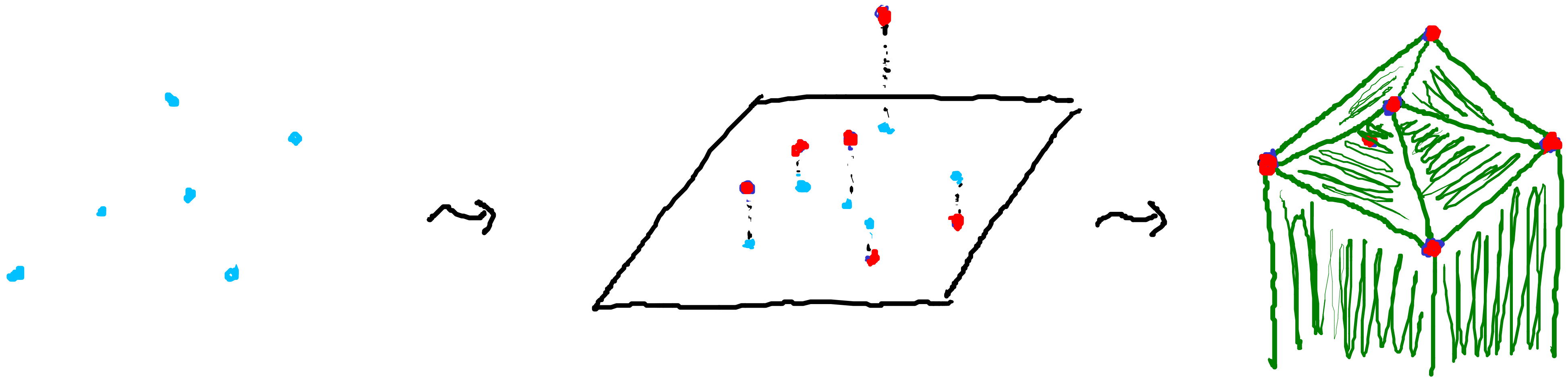
Regular (Mixed) Subdivisions?

$A \subseteq \mathbb{R}^n$

$\pi : A \to \mathbb{R}$ lifting

upper envelope of A wr π

$$\mathcal{L}(A, \pi) := \operatorname{conv}\left\{ \begin{pmatrix} \pi(\alpha) - s \\ \alpha \end{pmatrix} \;\middle|\; \alpha \in A,\ s \geq 0 \right\}$$



Our Result: In all its detail

$A_1, \dots, A_n \subseteq \mathbb{R}^n$ w/ $t_i = \# A_i$ fixed

$F := \{ F_i := \sum_{\alpha \in A_i} F_{i,\alpha} X^\alpha \quad (i = 1, \dots, n)$

s.t. the $F_{i,\alpha}$ are independent, centered & Gaussian

$\Pi_{F,i} : A_i \to \mathbb{R}$ given by $\Pi_{F,i}(\alpha) = \frac{1}{2} \ln \mathbb{V}(F_{i,\alpha})$

$$\mathop{\mathbb{E}}_F Z_r(F, \mathbb{R}^n_+) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n \mathcal{L}(A_i, \Pi_{F,i})\right) \prod_{i=1}^n (t_i - 1)$$

#vertices #Minkowski sum

⚠ #vertices of regular mixed subdivision induced by variance on supports

Three Tools

that made this possible

Tool I: Kac-Rice Formula

under some technical assumptions...

$$\mathbb{E}_f \, Z(f, \Omega) = \int_\Omega \mathbb{E}\left(|\det D_x f| \;\middle|\; f(x) = 0 \right) \delta_{f(x)}(0)\, dx$$

conditional expectation

density of $f(x)$

Tool II: Cauchy-Binet Formula

(aka the Fundamental Lemma of Fewnomial Theory)

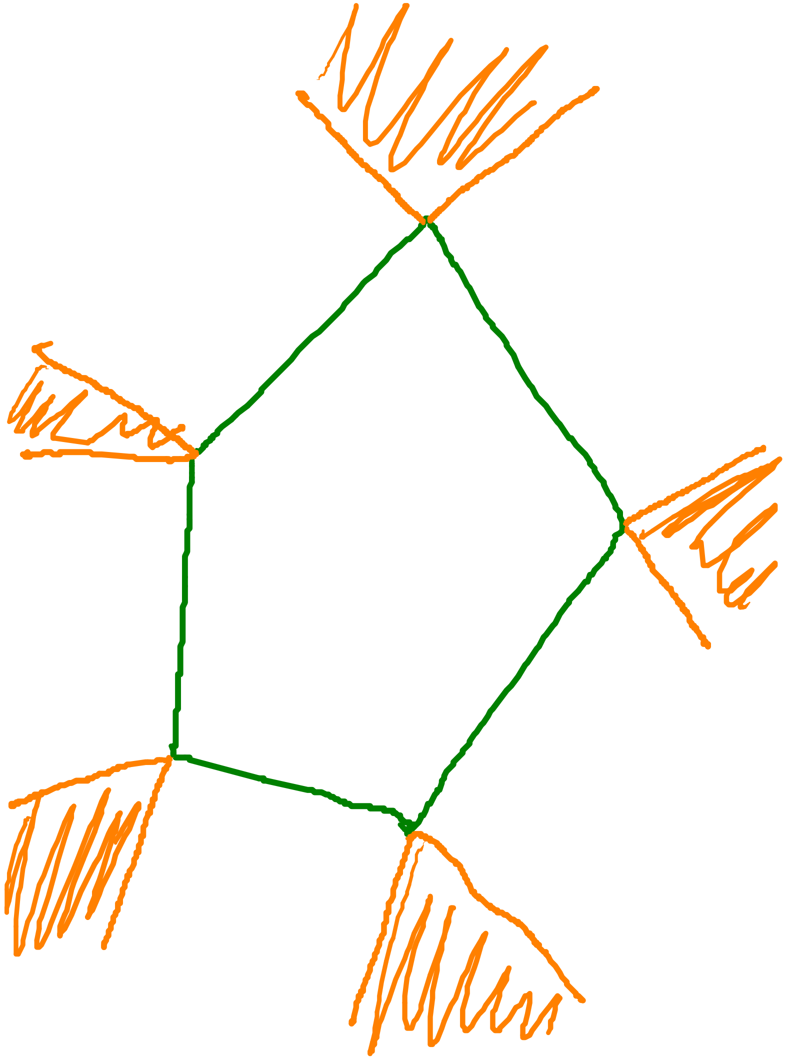
$A, B \in \mathbb{F}^{m \times n}$

$$\det(AB^T) = \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \# J = m}} \det(A_J)\det(B_J)$$

where $A_J = (A_{i,j})_{\substack{i \in \{1,\dots,m\} \\ j \in J}}$, $B_J := (B_{i,j})_{\substack{i \in \{1,\dots,m\} \\ j \in J}}$

used also in the work of Bihan & Sottile

Tool III: Normal Fan



$\rightsquigarrow$

Open Questions

What about higher moments?

In particular, what about

$$\bigvee_{F} Z_{r}(F, \mathbb{R}_{+}^{n}) \;?$$

What about more general Gaussian fewnomial systems?

Vielen Dank

für

Aufmerksamkeit!