Probabilistic Bounds on Best Rank-One Approximation Ratio

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Tensors I: Distances

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

space of tensors $\mathbb{K}^{n} := \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$

i.e. of multilinear maps $T : \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_d} \to \mathbb{K}$

of the form $T(x^1, \ldots, x^d) = \sum t_{i_1,\ldots,i_d} x^1_{i_1} \cdots x^d_{i_d}$

Frobenius product & norm

$$\langle T, T' \rangle := \sum \overline{t_{i_1,\ldots,i_d}} \, t'_{i_1,\ldots,i_d}$$

$$\|T\| := \sqrt{\langle T, T \rangle}$$

How to measure distances between tensors

Tensors II: Rank-one tensors

$x^i \in S(\mathbb{K}^{n_i}) \qquad \lambda \in \mathbb{K}$

$$\lambda x^1 \otimes \cdots \otimes x^d := \left(\lambda x^1_{i_1} \cdots x^d_{i_d}\right)$$

Easy to store! $1 + \sum n_i$ vs $\prod n_i$ numbers

THM. Given $T \in \mathbb{K}^{n_i}$, then there are $\lambda_1 x^1_1 \otimes \cdots \otimes x^d_1, \ldots, \lambda_r x^1_r \otimes \cdots \otimes x^d_r$ s.t.

$$T = \sum_{i=1}^{r} \lambda_i x^1_i \otimes \cdots \otimes x^d_i$$

minimum such $r =:$ rank T

Tensors III: Best Rank-One Approximation

$T \in \mathbb{K}^{\mathbf{n}}$ might not be rank-one

$$\text{Best Rank-One Approx}(T) := \underset{\substack{\lambda \in \mathbb{K} \\ x^i \in S(\mathbb{K}^{n_i})}}{\arg\min} \left\| T - \lambda\, x^1 \otimes \cdots \otimes x^d \right\|$$

Q: How good is this best rank-one approximation?

$$\min \left\| T - \lambda\, x^1 \otimes \cdots \otimes x^d \right\| = \|T\| \sqrt{1 - \left(\frac{\|T\|_\infty^{\mathbb{K}}}{\|T\|} \right)^2}$$

where $\|T\|_\infty = \max_{x^i \in S(\mathbb{K}^{n_i})} \left| T(x^1, \dots, x^d) \right|$

Tensors IV: Best Rank-One Approx. Ratio

$$A(\mathbb{K}^{n}) := \min_{T \in \mathbb{K}^{n}} \frac{\|T\|_{\infty}^{\mathbb{K}}}{\|T\|} \in [0,1]$$

Worst Relative Error of Best Rank-One Approx. $= \sqrt{1 - A(\mathbb{K}^{n})^{2}}$

Obs.

$$A(\mathbb{K}^{n}) \leq \mathbb{E}_{T \in \mathbb{K}^{n}} \frac{\|\Upsilon\|_{\infty}^{\mathbb{K}}}{\|\Upsilon\|}$$

Tensors V: Main Theorem

$d \geq 3 \qquad n_1, \ldots, n_d \geq 2$

$$\frac{1}{\sqrt{\min_i \prod_{j \neq i} n_j}} \leq A(\mathbb{K}^n) \leq \frac{32\sqrt{d \ln d}}{\sqrt{\min_i \prod_{j \neq i} n_j}}$$

But... this was known!

What's new?

Symmetric Tensors I

$$T \in (\mathbb{K}^n)^{\otimes d}$$

$$T \text{ sym} :\Leftrightarrow \forall \sigma \in \Sigma_n \ \forall x^i \in \mathbb{K}^n$$
$$T(x^1, \ldots, x^d) = T(x^{\sigma(1)}, \ldots, x^{\sigma(d)})$$

space of sym. tensors

$$\mathrm{Sym}^d(\mathbb{K}^n) \subseteq (\mathbb{K}^n)^{\otimes d}$$

Symmetric Tensors II:
Polynomials in disguise

$$T \in \mathrm{Sym}^d(\mathbb{K}^n) \longleftrightarrow f \in P_{d,n}$$

homogeneous
d deg
n var

$$\langle T, x \otimes \dots \otimes x \rangle = f(x)$$

Frob $$\langle f, g \rangle = \sum_\alpha \binom{d}{\alpha}^{-1} \bar{f}_\alpha g_\alpha$$ Weyl

$$\|f\|_\infty^{\mathbb{K}} = \max_{x \in S(\mathbb{K}^n)} |f(x)|$$ L^∞-norm

Sym. Tensors III: Rank-One Sym Tensors

$$\lambda\, x \otimes \cdots \otimes x \longleftrightarrow \lambda \left(\sum_{i=1}^{n} x_i X_i \right)^d$$

THM. Given $g \in P_{d,n}$, then there are rank-one sym tensors $\lambda_1 (\sum_j a_j^1 X_j)^d, \ldots, \lambda_r (\sum a_j^r X_j)^d$ s.t.

$$g = \sum_{i=1}^{r} \lambda_i \left(\sum_j a_j^i X_j \right)$$

min such r $=$ symrank g

⚠ symrank $\neq$ rank

Sym. Tensors IV:

Sym Best-One Approximation

$$\underset{\lambda \in \mathbb{K}\ x \in S(\mathbb{K}^n)}{\operatorname{argmin}} \left\| \mathcal{S} - \lambda \left(\sum_i x_i X_i \right)^d \right\|$$

$$\Big\downarrow \text{Error}$$

$$\min \left\| \mathcal{S} - \lambda \left(\sum_i x_i X_i \right)^d \right\| = \| \mathcal{S} \| \sqrt{1 - \left(\frac{\| \mathcal{S} \|_\infty^{\mathbb{K}}}{\| \mathcal{S} \|} \right)^2}$$

Sym. Tensors V:

Sym. Best Rank-One Approx. Ratio

$$A(\mathrm{Sym}^d(\mathbb{K}^n)) := \min \frac{\|f\|_\infty^{\mathbb{K}}}{\|f\|} \in [0,1]$$

Worst Relative Error of Sym. Best Rank-One Approx. $= \sqrt{1 - A(\mathrm{Sym}^d(\mathbb{K}^n))^2}$

Obs.

$$A(\mathrm{Sym}^d(\mathbb{K}^n)) \leq \mathop{\mathbb{E}}_{f \in P_{n,d}} \frac{\|f\|_\infty^{\mathbb{K}}}{\|f\|}$$

Sym. Tensors VI: Main Theorem $\mathbb{R}$.

$d \geq 3 \quad n \geq 2$

$$\max\left\{ \frac{1}{\sqrt{2}\,d} \binom{d+n-1}{n-1}^{-1/2}, \frac{1}{\sqrt{n^{d-1}}} \right\}$$

Comes from non-sym case

$$\wedge\|$$

$$\mathcal{A}\left(\mathrm{Sym}^d(\mathbb{R}^n)\right)$$

$$\wedge\|$$

$$24\sqrt{\frac{n \ln d}{\binom{d+n/2-1}{d}}}$$

Sym. Tensors VI: Main Theorem $\mathbb{C}$

$d \gtrsim 3 \quad n \geq 2$

$$\max\left\{ \binom{d+n-1}{n-1}^{-1/2}, \frac{1}{\sqrt{n^{d-1}}} \right\}$$

$$\wedge\!\!|$$

$$\mathcal{A}\left(\operatorname{Sym}^d(\mathbb{C}^n)\right)$$

$$\wedge\!\!|$$

$$36 \sqrt{\frac{n \ln d}{\binom{d+n-1}{d}}}$$

Sym Tensors VII: $n >> d$

$$\frac{1}{\sqrt{n^{d-1}}}$$

$$\wedge\!\!|$$

$$A\left(\mathrm{Sym}^d(\mathbb{K}^n)\right)$$

$$\wedge\!\!|$$

$$36\,\frac{\sqrt{d!\ln d}}{\sqrt{n^{d-1}}}$$

Sym. Tensors VIII $\quad d \gg n$ [Fix n, $d \to \infty$]

$$\sqrt{\frac{(n-1)!}{2^d\, d^{n-1}}}\left(1+O\left(\frac{1}{d}\right)\right) \qquad\qquad \sqrt{\frac{(n-1)!}{d^{n-1}}}\left(1+O\left(\frac{1}{d}\right)\right)$$

$$\wedge\!\!| \qquad\qquad\qquad\qquad \wedge\!\!|$$

$$A\left(\mathrm{Sym}^d(\mathbb{R}^n)\right) \qquad\qquad A\left(\mathrm{Sym}^d(\mathbb{C}^n)\right)$$

$$\wedge\!\!| \qquad\qquad\qquad\qquad \wedge\!\!|$$

$$48\sqrt{\frac{\left(\frac{n}{2}\right)!\,\ln d}{2^d\, d^{\frac{n}{2}-1}}}\left(1+O\left(\frac{1}{d}\right)\right) \qquad\qquad 36\sqrt{\frac{n!\,\ln d}{d^{n-1}}}\left(1+O\left(\frac{1}{d}\right)\right)$$

$$\left(\frac{n}{2}\right)! := \Gamma\left(\frac{n}{2}+1\right)$$

Sym Tensors IX: A special case

$d = 3$

$$\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n)) \leq O(n^{-0.584\ldots}) \quad (\text{Li \& Zhao, } 2020)$$

$$\leq O(n^{-1}) \quad (\text{Kozhasov \& T.-C., } 2022+)$$

We also have results

for partially sym. tensors

... aka multihom polynomials

How do we get our results? I

(real sym. case)

THM. Let $n \geq 2$ and $\mathcal{F}: S^{n-1} \to [0,\infty)$ a random Lipschitz function whose Lipschitz constant, $\mathrm{Lip}(\mathcal{F})$, satisfies for some $L > 1$,

$$\mathrm{Lip}(\mathcal{F}) \leq L \max_{x \in S^{n-1}} \mathcal{F}(x),$$

Then for $t > 0$,

$$\mathbb{P}_{\mathcal{F}}\left(\max_{x \in S^{n-1}} \mathcal{F}(x) \geq t \right) \leq \frac{e^{2n} L^{n-1}}{\sqrt{n-1}} \max_{x \in S^{n-1}} \mathbb{P}_{\mathcal{F}}\left(\mathcal{F}(x) \geq \frac{t}{2} \right)$$

How do we get our results? II

(real sym. case)

$$\mathcal{F}(x) = \frac{|f(x)|}{\|f\|}$$

Kellogg's Thm $\Rightarrow L = d$

$$\mathbb{P}_f\left(\frac{\|f\|_\infty}{\|f\|} \geq t\right) \leq \frac{e^{2n} d^{n-1}}{\sqrt{n-1}} \max_{x \in S^{n-1}} \mathbb{P}_f\left(\frac{|f(x)|}{\|f\|} \geq \frac{t}{2}\right)$$

For f Kostlan random polynomial, we can estimate!

How do we get our results? II

We reduce estimating tails
of the maximum of a random map
to estimating tails
of an arbitrary evaluation
of such random map

How do we get our results? IV

Evaluation of f at x

Prop. Let $P:\mathbb{R}^N \longrightarrow V$ be an orthogonal projection onto a k-dim subspace $V \subseteq \mathbb{R}^N$ and $x \in \mathbb{R}^N$ std Gaussian. Then for $t \geq 0$

$$\mathbb{P}_x\left(\frac{\|Px\|_2}{\|x\|_2} \geq t\right) \leq 2\exp\left(-\frac{Nt^2}{4e^{k+\frac{1}{cN}}}\right)$$

$\frac{\|Px\|_2}{\|x\|_2}$ subgaussian (bounds for exp)

How do we get our results? V

- Lower bounds follow from norm inequalities

- For some upper bounds, we need f random harmonic

Thank you for your attention!

Dziękuję za uwagę!

Any question?