Today, 130 years ago,

the Soviet mathematician

Aleksandr KHINCHIN

was born

Known for his contributions to probability theory

Some Lower Bounds on the Reach of an Algebraic Variety

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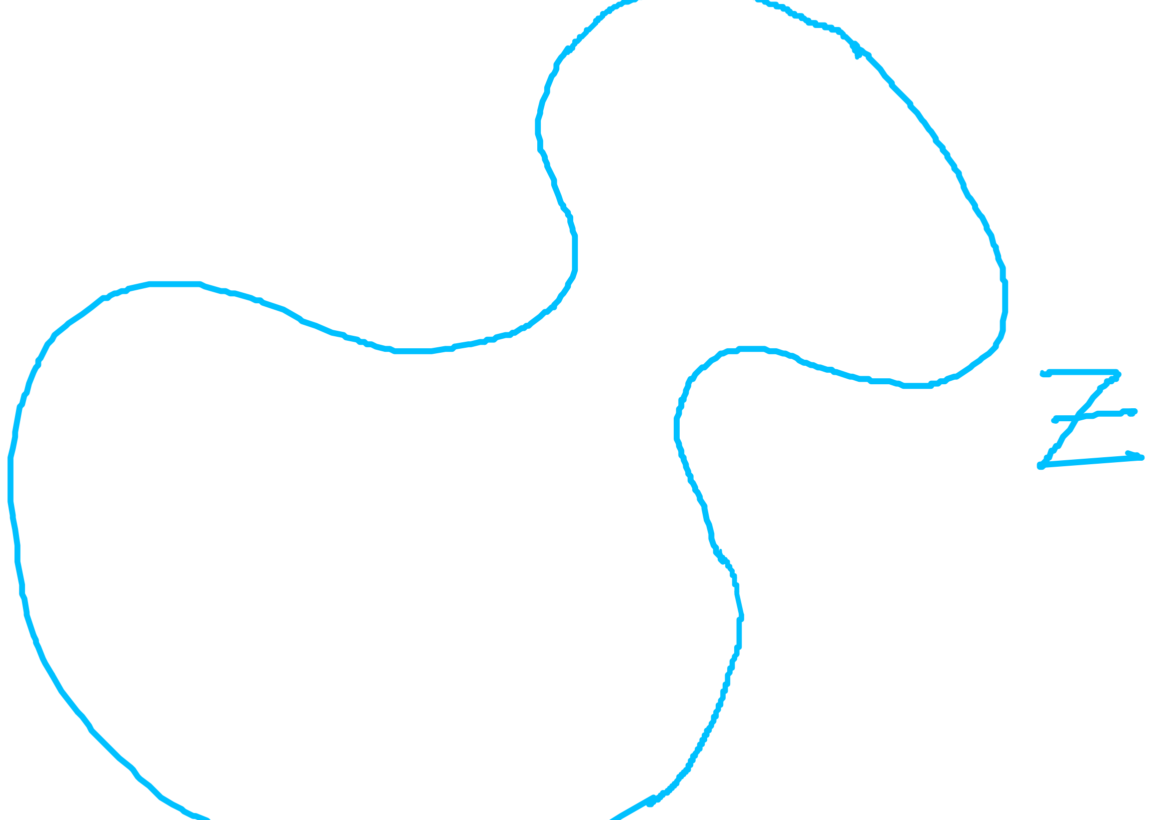
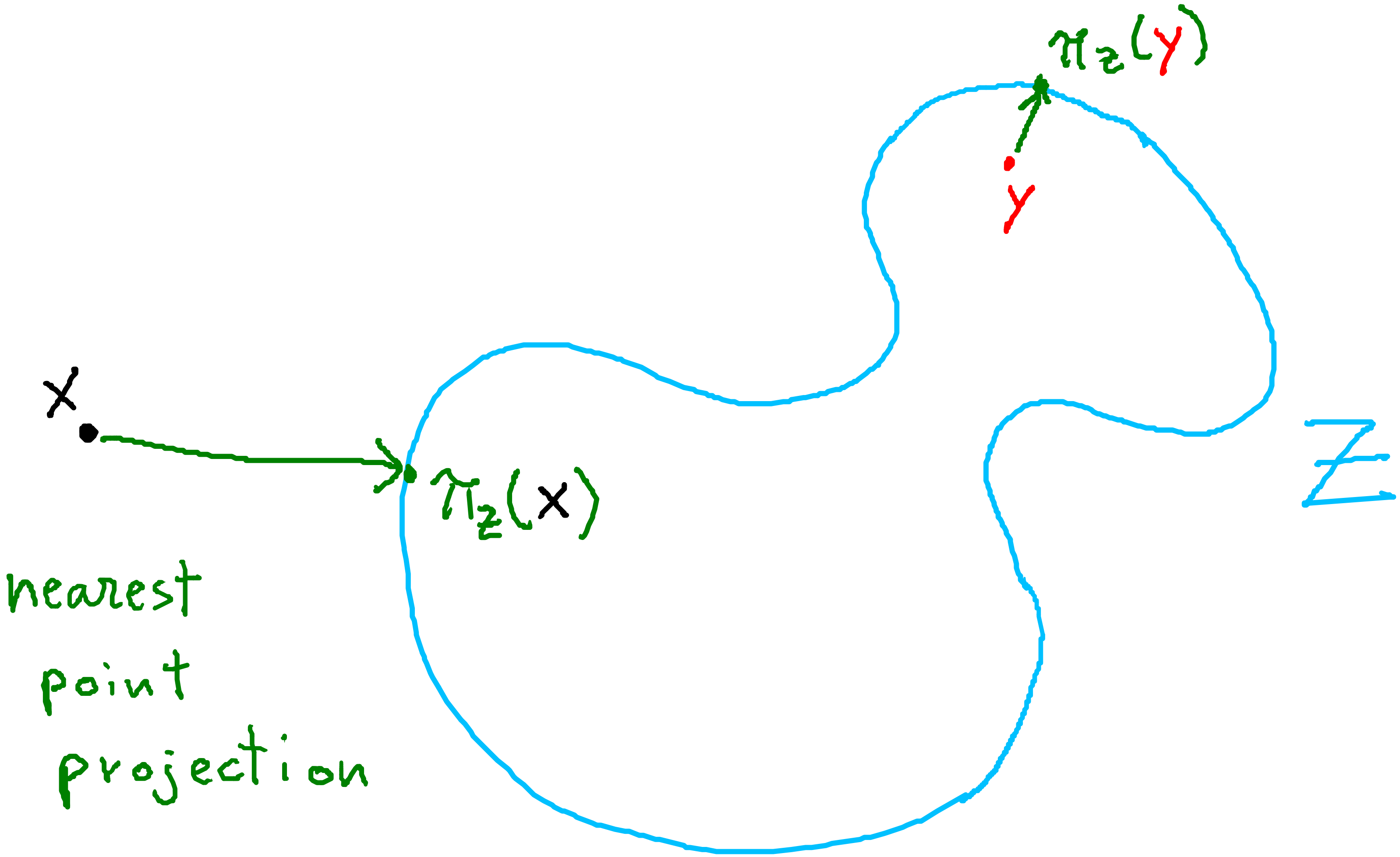
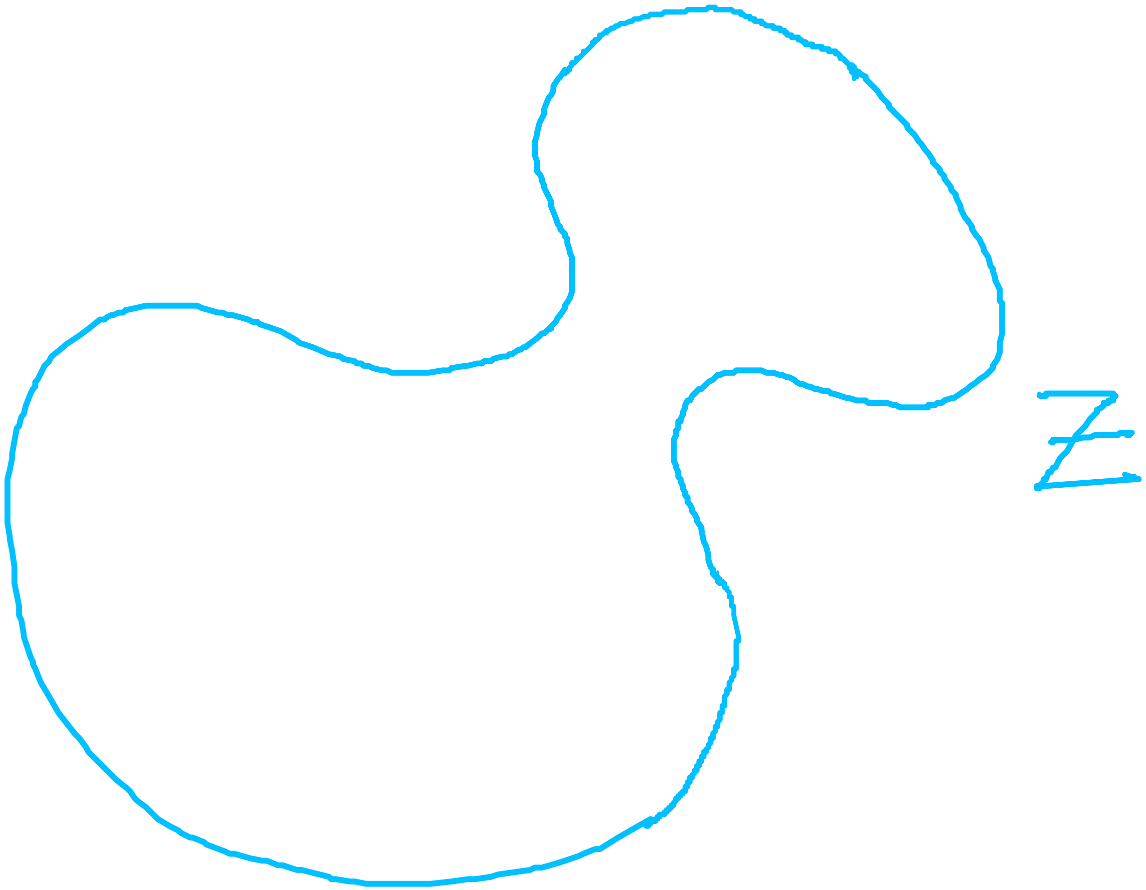
What is
the Reach?


Z



x
$\pi_Z(x)$
$\pi_Z(y)$
y
Z
nearest point projection

What if there is not only one nearest point?



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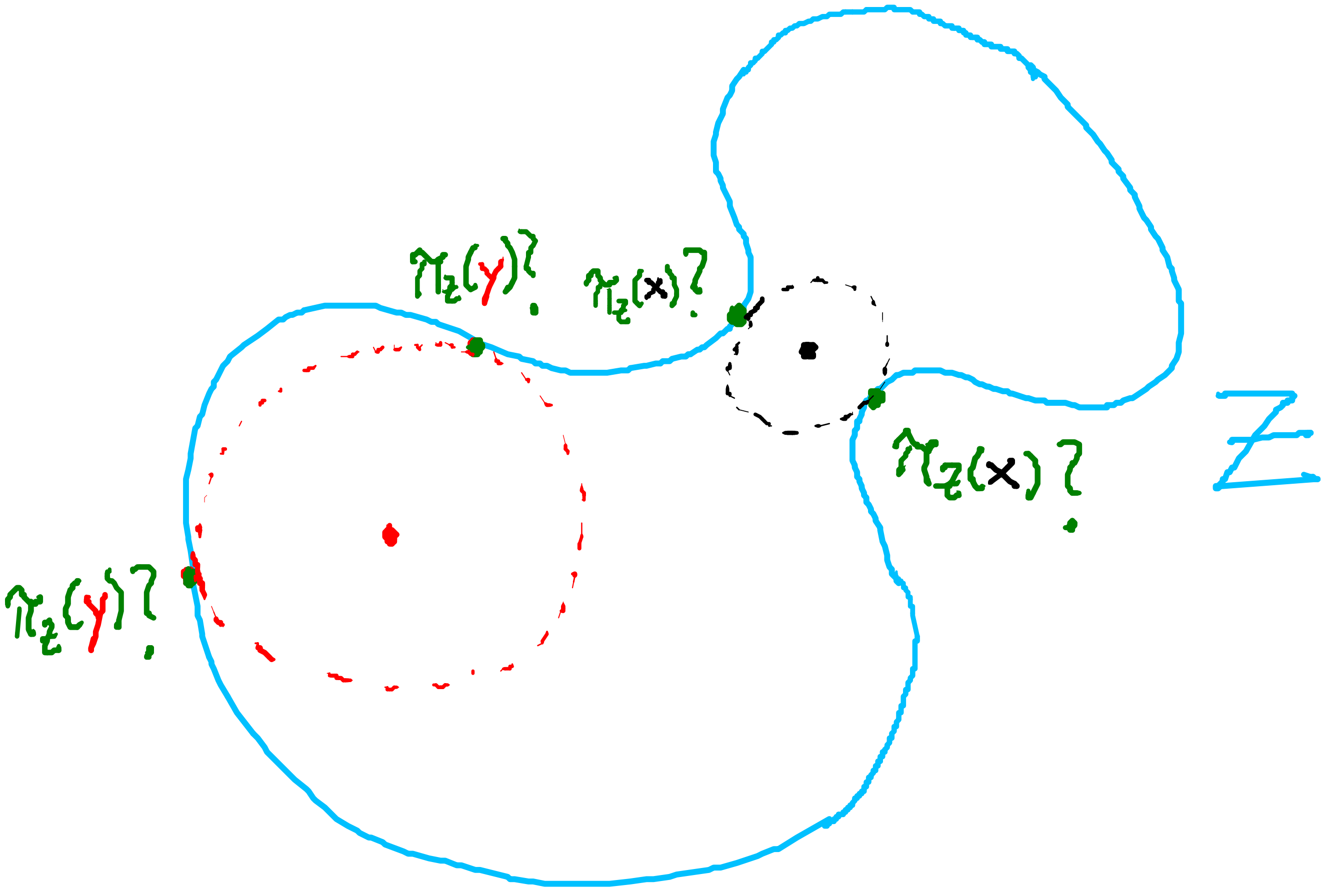

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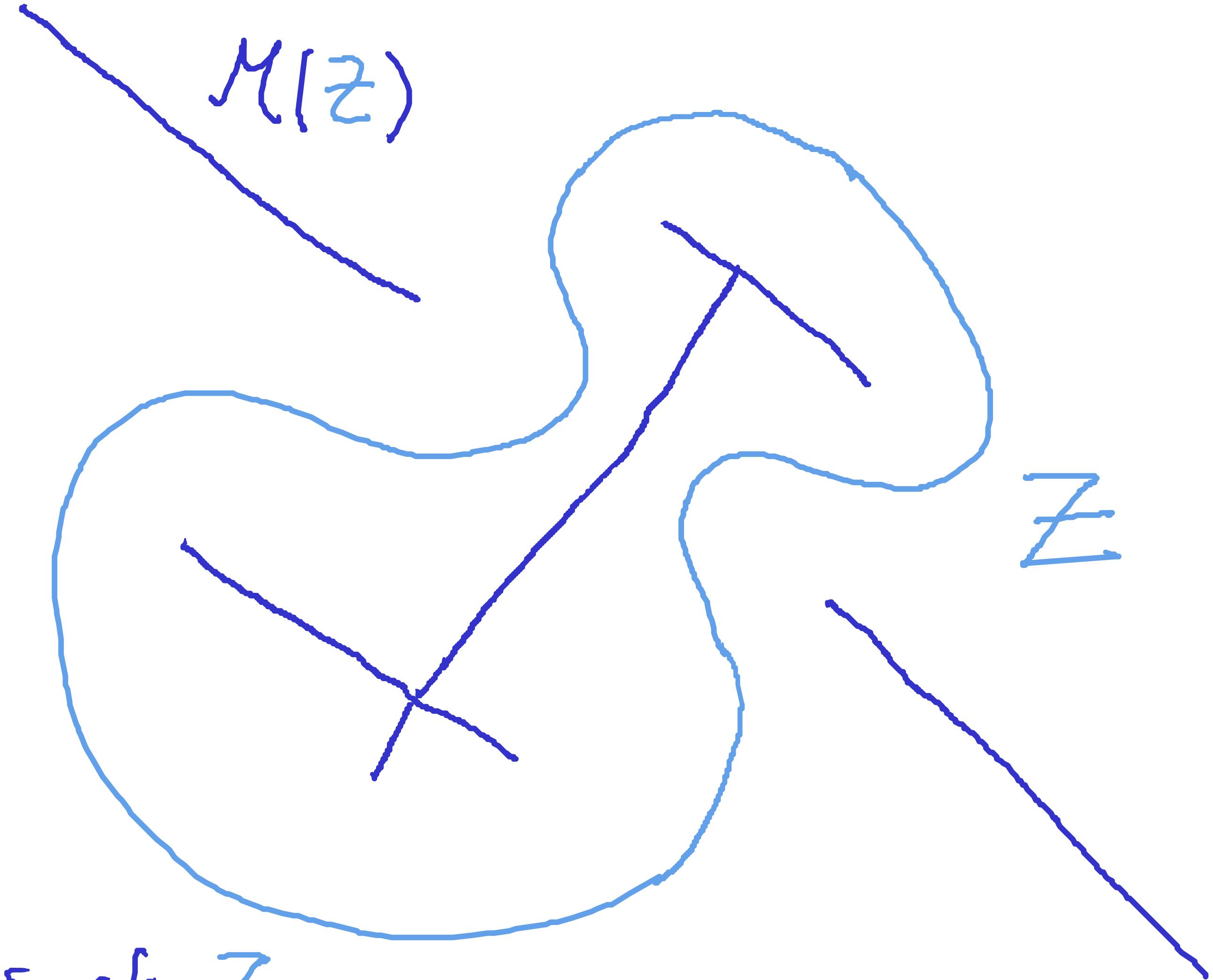

Medial Axis of Z

$$M(Z) := \{ x \mid \exists\, \xi, \tilde{\xi} \in Z;\ \xi \neq \tilde{\xi},\ \mathrm{dist}(x,\xi) = \mathrm{dist}(x,\tilde{\xi}) = \mathrm{dist}(x, Z) \}$$

Local Reach:

$$r(z, g) := \mathrm{dist}(g, \mathcal{M}(z))$$

Reach:

$$r(z) := \min_{g \in z} r(z, g) = \mathrm{dist}(z, \mathcal{M}(z))$$

And...

why do we care

about the reach?

TDA

Topological Data Analysis

Reach

is

the positive-dimensional

separation bound

Our Results

in a friendly version

Integer Worst Case:

$R \in \mathbb{N} \quad g \in \mathbb{Z}[X_1,\dots,X_n]^q_{\leq D} \qquad |g_{i,\alpha}| \leq 2^\tau$

$$c_R(\mathcal{Z}(g)) = 0 \quad (\mathcal{Z}(g) \text{ singular})$$

OR

$$\log \frac{1}{c_R(\mathcal{Z}(g))} \leq \mathcal{O}\left(n(2D)^{q+2n}\right)(\tau + \log R + n \log D)$$

$$c_R(Z) := \min\{c(Z,g) \mid g \in Z,\ \max|g_i| \leq R\}$$

$$c(Z,g) := \operatorname{dist}(g, \mathcal{M}(Z)) \qquad \mathcal{M}(Z): \text{Medial Axis of } Z$$

Integer Probabilistic Case

$R \in \mathbb{N} \quad f \in \mathbb{Z}[X_1,\dots,X_n]^q_{\le D}$

$f_{i,\alpha} \overset{\text{iid}}{\sim} \mathcal{U}(\mathbb{Z} \cap [-2^\tau, 2^\tau])$ ← Also for random bit polynomials

$$\log \frac{1}{\varrho_R(\mathcal{Z}(f))} \le \mathcal{O}(\log R + n \log n + (q+n) n \log D) + s$$

with prob. $\ge 1 - 2^{-s}$

for $s \le \mathcal{O}(\tau)$ and $\tau = \Omega(\log R + n\log n + (q+n)n\log D)$

$\varrho_R(\mathcal{Z}) := \min\{\varrho(\mathcal{Z}, g) \mid g \in \mathcal{Z}, \max |g_i| \le R\}$

$\varrho(\mathcal{Z}, g) := \operatorname{dist}(g, \mathcal{M}(\mathcal{Z}))$ $\quad \mathcal{M}(\mathcal{Z})$: Medial Axis of $\mathcal{Z}$

Continuous Probabilistic Case

$R \in \mathbb{R}_+ \quad F \in \mathbb{R}[X_1, \dots, X_n]^q_{\leq D}$

$F_{i,\alpha} \overset{iid}{\sim} \mathcal{U}([-1,1])$ ← Also for zintzo random polynomials!

$$\log \frac{1}{e_R(\mathcal{Z}(F))} \leq \mathcal{O}(\log R + n \log n + (q+n) n \log D) + s/2$$

with prob. $\geq 1 - 2^{-s}$

$e_R(Z) := \min\{e(Z,g) \mid g \in Z, \max|g_i| \leq R\}$

$e(Z,g) := \mathrm{dist}(g, \mathcal{M}(Z))$ $\quad \mathcal{M}(Z)$: Medial Axis of Z

Our Techniques

Federer's Lower Bound

$\mathcal{Z} \subseteq \mathbb{R}^m$ closed

$g \in \mathcal{Z}$

$r, t > 0$

IF $\mathcal{Z} \cap B(g, r) \subseteq B(g, r)$ closed submanifold

& for all $z, \tilde{z} \in \mathcal{Z} \cap B(g, r)$,

$$\operatorname{dist}(\tilde{z} - z, T_z \mathcal{Z}) \leq \frac{\|\hat{z} - z\|^2}{2t}$$

THEN:

$$e(z, g) \geq \min\{r, t\}$$

Improvement of Thm 3.3 of (Bürgisser, Cucker, Lairez; 2019)

$$\varrho(Z(g),g) \geq \frac{1}{5\,\gamma(g,g)}$$

instead of 14

where

$$\gamma(g,g) := \sup_{k\geq 2} \left\| D_g g^{\dagger} \frac{1}{k!} D_x^k g \right\|^{\frac{1}{k-1}}$$

is Smale's γ.

Condition Number

$$\|g\|_1 := \max_i \sum_\alpha |g_{i,\alpha}|$$

$$C(g, x) \propto \frac{\|g\|_1}{\mathrm{dist}_1(g, \Sigma_x)}$$

$$\Sigma_x := \{ g \mid x \in Z(g) \text{ singular} \}$$

Copernican turn:

We don't ask about the quality of the zero, but about the quality of the system around the zero.

A new condition-based bound

$$\mathcal{C}(Z(\delta), g) \geq \frac{\max_i \{1, |g_i|\}}{\max\{D-2, \mathcal{C}(\delta, g)\}}$$

$$C(g, x) = \sup_{v} \frac{\|g\|_1}{\|\mathcal{L}_{x,v}\, g\|}$$

linear projection
polynomically depending on x
and linearly on v

Geometric

Functional Analysis

- Anti-concentration of linear projections

x anticoncentrated $\Rightarrow$ Ax anti-concentrated
+ ind. comp.

- Ball's smoothing

x disc. $\Rightarrow$ $x+y$ cont. (+ details)
y cont.
well-chosen

anti concentrated

Thank You

for your Attention!