Ultrametric Smale's α-Theory

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Newton's Method

$$N_f : x \mapsto x - D_x f^{-1} f(x)$$

$$N_f^{k+1}(x) := N_f(N_f^k(x))$$

$$N_f^k(x) \xrightarrow[k\to\infty]{} \text{zero of } f$$

...but not always

Question:

When does Newton's method converge fast to a zero?

Case of Interest:

$f \in \mathbb{F}[X_1,\dots,X_n]^n$ polynomial system

x approximate zero

When is x a 'good' approximate zero of f?

Real/Complex Smale's α-Theory (the arquimedian case)

Smale's α: $\alpha(f,x) := \beta(f,x)\gamma(f,x)$

Smale's β: $\beta(f,x) := \| D_x f^{-1} f(x) \| = \operatorname{dist}(x, N_f(x))$ ← Length of Newton step

Smale's γ: $\gamma(f,x) := \sup_{k\geq 2} \left\| D_x f^{-1} \frac{1}{k!} D_x^k f \right\|^{\frac{1}{k-1}}$ ← "A measure of analyticity"

Smale's γ-Theorem

ζ a zero of f

$\gamma(f,\zeta)\operatorname{dist}(x,\zeta) < \gamma_*$ ← Absolute constant

$\Rightarrow N_f^k(x) \xrightarrow[k\to\infty]{} \zeta$ fast

Criterion in terms of the zero:
How near I have to be of the zero to guarantee fast convergence?

Smale's α-Theorem

$\alpha(f,x) < \alpha_* \Leftrightarrow \beta(f,x) < \alpha_*/\gamma(f,x)$ ← Absolute constant

$\Rightarrow N_f^k(x) \xrightarrow[k\to\infty]{} \text{zero of } f$ fast

Criterion only in terms of initial point:
How can I guarantee fast convergence without knowing the zero?
How small has to be the first Newton step to get fast convergence?

WHAT DOES 'FAST' MEAN?

$$\operatorname{dist}(N_f^k(x), \zeta) < \beta_*/2^{2^k-1}$$

Each iteration doubles the precision of the approximation!

p-adic Smale's α-Theory (the char 0 non-arquimedian case)

Main case of interest: $\mathbb{Q}_p$
Newton's method ⇔ Hensel's lifting

α, β, γ the same, but two theorems in one!

Smale's α/γ-Theorem

Easy constants!

$\alpha(f,x) < 1 \Leftrightarrow \gamma(f,x)\operatorname{dist}(x, \text{zeroes of } f) < 1$ ← γ-criterion in disguise in the ultrametric setting

$\Rightarrow N_f^k(x) \xrightarrow[k\to\infty]{} \text{zero of } f$ fast

If $n=1$, this implies Hensel's lifting!