# Ultrametric Smale's $\alpha$-Theory

Jazz G. Suchen
IIS Berlin

Josué Tonelli-Cueto
Inria Paris & IMJ-PRG

## Newton's Method

$$N_f : x \longmapsto x - D_x f^{-1} f(x)$$

$$N_f^{k+1}(x) := N_f\left(N_f^k(x)\right)$$

$N_f^k(x) \xrightarrow[k\to\infty]{}$ zero of $f$

...but not always

## Question:

When does Newton's method converge fast to a zero?

## Case of Interest:

$f \in \mathbb{F}[X_1,\dots,X_n]^n$ polynomial system

$x$ approximate zero

When is $x$ a 'good' approximate zero of $f$?

## Real/Complex Smale's $\alpha$-Theory (the arquimedian case)

Smale's $\alpha$: $\alpha(f,x) := \beta(f,x)\gamma(f,x)$

Smale's $\beta$: $\beta(f,x) := \|D_x f^{-1} f(x)\| = \operatorname{dist}(x, N_f(x))$ ← Length of Newton step

Smale's $\gamma$: $\gamma(f,x) := \sup_{k\geq 2} \left\|D_x f^{-1} \frac{1}{k!} D_x^k f\right\|^{\frac{1}{k-1}}$ ← "A measure of analyticity"

## Smale's $\gamma$-Theorem

$\zeta_f$ a zero of $f$ ← Absolute constant

$$\gamma(f,\zeta_f)\operatorname{dist}(x,\zeta_f) < \gamma_* \Rightarrow N_f^k(x) \xrightarrow[k\to\infty]{} \zeta_f \text{ fast}$$

Criterion in terms of the zero:
How near I have to be of the zero to guarantee fast convergence?

⚠ WHAT DOES 'FAST' MEAN?

$$\operatorname{dist}\left(N_f^k(x), \zeta_f\right) < \beta_* / 2^{2^k - 1}$$

Each iteration doubles the precision of the approximation!

## Smale's $\alpha$-Theorem

Absolute constant →

$$\alpha(f,x) < \alpha_* \Leftrightarrow \beta(f,x) < \alpha_*/\gamma(f,x) \Rightarrow N_f^k(x) \xrightarrow[k\to\infty]{} \text{zero of } f \text{ fast}$$

Criterion only in terms of initial point:
How can I guarantee fast convergence without knowing the zero?
How small has to be the first Newton step to get fast convergence?

## p-adic Smale's $\alpha$-Theory (the char 0 non-arquimedian case)

$\alpha, \beta, \gamma$ the same, but two theorems in one!

↳ Main case of interest: $\mathbb{Q}_p$

Newton's method ⇔ Hensel's lifting

### Smale's $\alpha/\gamma$-Theorem

Easy constants! ↷

$$\alpha(f,x) < 1 \Leftrightarrow \gamma(f,x)\operatorname{dist}(x, \text{zeroes of } f) < 1 \Rightarrow N_f^k(x) \xrightarrow[k\to\infty]{} \text{zero of } f \text{ fast}$$

← $\gamma$-criterion in disguise in the ultrametric setting

If $n = 1$,
this implies Hensel's lifting!