

Metric Restrictions

on the

Number

of

Real Zeros

Joint work with
Elias Tsigaridas

Josué TONELLI-CUETO
Inria Paris & IMJ-PRG

As some ideas are less technical
in the setting of homogeneous polynomials,
I will focus on that setting.

Symmetry makes life easier.

NOTATIONS

• X_0, X_1, \dots, X_n homogeneous variables

• $\mathcal{H}_d[n] := \prod_{i=1}^n \mathbb{R}[X_0, \dots, X_n]_{d_i}$

• For $f \in \mathcal{H}_d[n]$, and $x \in S^n$

$$\bar{D}_x^k f = \left(\frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right)$$

$$D_x f = \bar{D}_x f (\mathbb{I} - xx^t) = \bar{D}_x f |_{T_x S^n}$$

$$\text{Weyl norm } \|f\|_W := \sqrt{\sum_{i=1}^n \sum_{|\alpha|=d_i} \binom{d_i}{\alpha}^{-1} |f_{i,\alpha}|^2}$$

$$\mathcal{Z}_S(f) := \{x \in S^n \mid f(x) = 0\}$$

DISCRIMINANT CHAMBERS

$$\Sigma := \{g \in \mathcal{H}_d[n] \mid \mathcal{Z}_S(g) \text{ singular}\}$$

↑ Here is where changes can occur!

Prop. $\mathcal{H}_d[n] \setminus \Sigma \ni g \mapsto \# \mathcal{Z}_S(g)$ locally constant.

Def. A discriminant chamber \mathcal{A} is a connected component of $\mathcal{H}_d[n] \setminus \Sigma$

Question. Given $f \in \mathcal{H}_d[n]$ KSS* random polynomial system, what is $P(g \in \mathcal{A})$?

*Also for dobro

INTERLUDE: RANDOM POLYNOMIAL SYSTEM

Let $f \in \mathcal{P}_d[n]$ with

$$f_i = \sum_{\alpha} \sqrt{\binom{d_i}{\alpha}} c_{i,\alpha} X^{\alpha}$$

be random.

- **KSS**: $c_{i,\alpha}$ i.i.d. normal
- **Dobro**: $c_{i,\alpha}$ independent, anti-conc + subgaussian
- **EPR**: f_i independent, $f_i(x)$ anti-conc. + subgauss.

Evgün, Paouris
Rojas

CONDITION NUMBER

Def. Given $g \in \mathcal{H}_d[n]$, the condition number of g is

$$\kappa(g) := \max_{x \in S^n} \frac{\|g\|_w}{\sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta\|^{-1}}} \in [1, \infty]$$

where $\Delta := \text{diag}(d_1, \dots, d_n)$

THM (Condition Number Theorem) [Cucker, Kyrick, Malajovich, Wschebor]

Let $g \in \mathcal{H}_d[n]$, then

$$\kappa(g) = \frac{\|g\|_w}{\text{dist}_w(g, \Sigma)}$$

κ is a metric discriminant!

INRADIUS OF A DISC. CHAMBER

DEF. Let $A \subseteq \mathcal{H}_d(h) \setminus \Sigma$ be a disc chamber, and consider

$$\kappa(A) := \min \{ \kappa(g) \mid g \in A \}$$

OBS. Given $g \in A$,

$$g + \|g\|_w \vee B_w \subseteq A \iff \text{dist}(g, \Sigma) \geq \|g\|_w \vee$$

PROP. $1/\kappa(A)$ is the inradius of A ,

$$\text{i.e., } 1/\kappa(A) = \max \{ r \mid \exists g \in A : B_w(g, r\|g\|_w) \subseteq A \}$$

↑
 A is conic!

$$B_w := \{ g \mid \|g\|_w < 1 \}, B_w(g, s) := \{ h \mid \|h - g\|_w < s \}$$

BOUNDING PROBABILITIES I

Let $f \in \mathcal{H}_d[n]$ be a random KSS pol. system,

then:

$$\mathbb{P}(f \in \mathcal{A}) \leq \mathbb{P}(\kappa(f) \geq \kappa(\mathcal{A}))$$

$$\leq 32 D^2 D^{1/2} N^{1/2} n^3 \frac{\ln^{1/2} \kappa(\mathcal{A})}{\kappa(\mathcal{A})}$$

↑
Cucker, Krick

Malajovich, Wschebor

↑
Can we lower bound $\kappa(\mathcal{A})$?

Yes!

The above works for very general random assumptions...

AN UNEXPECTED INEQUALITY

THM. Let $f \in \mathcal{H}_d[n]$. Then

$$\# \mathcal{Z}_S(f) \leq c^n D^{n/2} \log^n K(f)$$

where $c \geq 1$ is universal.

COR. Let $f \in \mathcal{H}_d[n]$. Then

$$K(f) \geq 2 \frac{\# \mathcal{Z}_S(f)^{1/n}}{c D^{n/2}}$$

COR. If $f \in \mathcal{H}_d[n]$ has many real zeros, then f is ill-conditioned!

BOUNDING PROBABILITIES II

THM. Let $\mathcal{A} \subseteq \mathcal{H}_d[n]/\Sigma$ be a discriminant chamber and $N(\mathcal{A})$ the number of real zeros of any system in \mathcal{A} , then:

$$\mathbb{P}(F \in \mathcal{A}) \leq 2^{a \log D - b \frac{N(\mathcal{A})^{1/n}}{D^{1/2}}}$$

where $a, b > 0$ are universal.

COR. Disc. chambers with systems with many real zeros are always small.

BOUNDING PROBABILITIES III

THM. Let f be a KSS random polynomial system, then

$$\# \mathcal{Z}_S(f)^{1/n}$$

is subexponential with constant

$$a n D^{1/2} \log D$$

where $a > 0$ is universal. I.e. for $e \geq 1$,

$$\left(\mathbb{E} \mathcal{Z}_S(f)^e \right)^{1/e} \leq a^n n^{2n} D^{n/2} \log^n D e^n$$

WHAT'S BEHIND

THE UNEXPECTED INEQUALITY?

(Moroz 2021)

To solve a univariate polynomial uses many extremely low degree approximations based on Taylor expansions.

We generalize this to higher dimensions

THM. Let $g \in \mathcal{H}_d[n]$, and $r < 1/d^{1/2}$. Then for all $x \in S^n$, $\mathcal{H}_{B_S}(x, r)$ can be approximated by a $O(\log K(g))$ -degree pol. system with zeros that approximate à la Smale all those of g in $B_S(x, r)$

OBS. This many extremely low-degree approx. scheme differs from the one low-degree of Diatta & Levario.

OTHER CASES...

- Kac random polynomial systems
- Underdetermined polynomial systems
(only volume for now)
- Sparse Kac random polynomial systems
(we have to see how strong
can our results be)

FUTURE

Can we have algorithms
working in time that is
bounded by

$$\log^n K(\delta)$$

and not $K(\delta)^n$?

This should give very fast
algorithms in NREG

Main obstacle: Avoid computing $K(\delta)$ directly...

•
¡Muchas

gracias

por su atención!