How many $\mathbb{R}$eal Zeros does a random sparse polynomial system have?

Algebra Seminar
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How many... Complex Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

d complex zeros (generically)

How many... Real Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

≤ 2 real zeros (generically)

How many...

Positive Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

0 or 1 positive zeros (generically)

General Phenomenon...

Many $\mathbb{C}$omplex Zeros,

but a lot fewer $\mathbb{R}$eal Zeros

Why do we care about Real Zeros?

- Chemical Reaction Networks
- Computer Vision
- Phylogenetic Trees

[Applied answer]

Why do we care about $\mathbb{R}$eal Zeros?

In applications,
the $\mathbb{R}$eal & $\mathbb{P}$ositive Zeros
are the ones that matter

[Applied Answer]

Why do we care about $\mathbb{R}$eal Zeros?

The number of $\mathbb{C}$omplex zeros

is a geometric quantity:

degree, (mixed) volume of Newton polytopes

The number of $\mathbb{R}$eal zeros

is a more combinatorial quantity:

size of the support

[Pure answer]

Khovanskii's Theorem on Fewnomial Systems

Kii's Theorem

Fewnomial Systems

¿? →

Polynomial → poly-nomial → много-член

→ многочлен → малочлен

"many monomial" "few monomials"

→ мало-член → few-nomial → Fewnomial

⇝ Gutxinomioa ⇝ Poconomio

Khovanskii's Theorem on Fewnomial Systems

$f \in \mathbb{R}[X_1, \ldots, X_n]^n$ fewnomial system with at most t exponents

$$\# \mathcal{Z}_r(f, \mathbb{R}^n_+) \leq 2^{\binom{t-1}{2}} (n+1)^{t-1}$$

No degree!

where $\mathcal{Z}_r(f, \mathbb{R}^n_+) := \{ x \in \mathbb{R}^n_+ \mid f(x) = 0,\ D_x f \text{ non-sing.} \}$

Kushnirenko's Hypothesis

$g \in \mathbb{R}[x_1, \ldots, x_n]^n$ fewnomial system

where g_i has t_i terms

Is there universal $c > 0$ s.t.

$$\# Z_r(g, \mathbb{R}_+^n) \leq (t_1 \cdots t_n)^c \ ?$$

State of the Art (Zero dimensional case)

(Descartes, XVII) if $n=1$, $\# Z(g, \mathbb{R}_+) \leq t-1$

(Khovanskii, 1991)

$$\# Z_r(g, \mathbb{R}^n_+) \leq 2^{\binom{t-1}{2}} (n+1)^{t-1}$$

(Bihan, Sottile; 2007)

$$\# Z_r(g, \mathbb{R}^n_+) \leq 3 \cdot 2^{\binom{t-n-1}{2}} \cdot n^{t-n-1}$$

(Sevastyanov, 1978; Koiran, Portier, Tavenas; 2015)

if $n=2$, $\# Z_r(g, \mathbb{R}^2_+) \leq O(d_2^3 t_1 + d_2^2 t_1^3)$

State of the Art (Zero dimensional case)

Even the bivariate case $(n = 2)$

is widely open!

A Probabilistic Approach

Randomize F,

what can we say about

$$\mathbb{E}\#Z_r(F, \mathbb{R}^n_+)?$$

Probabilistic State of the Art I

(Bügisser, Ergür, Tonelli-Cueto; 2018)

$f_i = \sum_{\alpha \in A} f_{i,\alpha} X^{\alpha}$ with $f_{i,\alpha}$ ind. cent. Gaussian

s.t. $\mathrm{Var}(f_{i,\alpha})$ only depends on α

$t := \# A$

$$\mathbb{E}\, \# Z_r(f, \mathbb{R}^n_+) \leq \frac{1}{2^{n-1}} \binom{t}{n}$$

Probabilistic State of the Art II

(Bürgisser, ISSAC'2023)

$f_i = \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha$ with $f_{i,\alpha}$ i.i.d. standard Gaussian

$t_i := \# A_i$

$$\mathbb{E}\, \# Z_r(f, \mathbb{R}^n_+) \leq \frac{1}{(2\pi)^{\frac{n}{2}}} V\left(\sum_{i=1}^n P_i\right) t_1 \cdots t_n$$

where $P_i := \mathrm{conv}(A_i)$ & $V(P) := \#$ vertices of P

Main Result I (Ergür, Telek, Tonelli-Cueto; 2023+)

$f \in \mathbb{R}[X_1, \dots, X_n]^n$ random fewnomial system such that the coefficients are ind. cent. Gaussians

$$\mathbb{E}\,\#\mathcal{Z}_r(f, \mathbb{R}^n_+) \leq \frac{1}{4^n}\, t_1(t_1-1)\cdots t_n(t_n-1)$$

where t_i is # monomials of f_i

Main Result II (Ergür, Telek, Tonelli-Cueto; 2023+)

$$f_i = \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha$$ with $f_{i,\alpha}$ ind. cent. Gaussians

$t_i := \# A_i$

If $\mathrm{Var}(f_{i,\alpha}) \leq 1$,
with equality whenever α is a vertex P_i

$$\mathbb{E}\, \# Z_r(f, \mathbb{R}^n_+) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n P_i\right) t_1 \cdots t_n$$

where $P_i := \mathrm{conv}(A_i)$ & $V(P) := \#$ vertices of P

Main Result III (Ergür, Telek, Tonelli-Cueto; 2023+)

$f_i = \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha$ with $f_{i,\alpha}$ ind. cent. Gaussians

$t_i := \# A_i \quad \Lambda_i : A_i \to \mathbb{R}$ s.t. $\Lambda_i(\alpha) := \log \operatorname{Var}(f_{i,\alpha})$

$\mathcal{L}(A, \Lambda) := \operatorname{conv}\left\{ \left({}^{\Lambda(\alpha) - s}_{\alpha} \right) \mid \alpha \in A \quad s \geq 0 \right\}$

$$\mathbb{E}\, \# Z_r(f, \mathbb{R}^n_+) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n \mathcal{L}(A_i, \Lambda_i) \right) t_1 \cdots t_n$$

where $V(P) := \#$ vertices of P

Two Tools I: Rice's Formula

$$\mathbb{E}\,\#Z(f,\mathbb{R}^n_+)=\int_{\mathbb{R}^n_+}\mathbb{E}\left(|\det D_x f|\;\middle|\;f(x)=0\right)\rho_{f(x)}(0)\,dx$$

Two Tools II: Cauchy-Binet Formula

A, B $m \times n$ matrices

$$\det AB^T = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I = m}} \det A_I \det B_I$$

where X_I is the submatrix of X

· whose columns are indexed by I

What about the variance?

THM (Tsigaridas, Tonelli-Cueto; 2024+)

$f \in \mathbb{R}[X_1, \dots, X_n]^n$ random fewnomial system
with $\sum P_i$ simple, where $P_i = \operatorname{conv}(A_i)$
& A_i support of A_i

$$\left(\mathbb{E}\, \#\mathcal{Z}_r(f, \mathbb{R}^n_+)^e\right)^{1/e} \leq V\left(\sum_{i=1}^{n} P_i\right) \mathcal{O}(\log^2 D + n \log D)^n e^n$$

where $V(P) := \#$ vertices of P & $D =$ max degree

Future Work:

Can we prove Kashnirenko's Hypothesis III probabilistically?

Eskerrik
Asko!