Computing the homology of closed semialgebraic sets

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I: Lax Formulas

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Semialgebraic sets I

Definition

Let f be a tuple of polynomials, a *(semialgebraic) formula* Φ over f is a Boolean formula (using negation $\neg$, conjunction $\wedge$ and disjunction $\vee$) formed from atoms of the form $(f_i < 0)$, $(f_i \leq 0)$, $(f_i = 0)$, $(f_i > 0)$, $(f_i \geq 0)$ and $(f_i \neq 0)$.

The semialgebraic set $W(f, \Phi)$ is the set obtained from Φ by interpreting the atoms in the usual way, negations as complements, conjunctions as intersections and disjunctions as unions.

Example

1. Full circle with a tangent line: $(x^2 + y^2 - 1 \leq 0) \vee (y - 1 = 0)$.
2. Two lines crossing: $(3x + 2y = 0) \vee (7x - 4y = 0)$.
3. A line intersection with the positive and negative orthants: $(3x - y = 0) \wedge (((x > 0) \wedge (y > 0)) \vee ((x < 0) \wedge (y < 0)))$.

Semialgebraic sets II

Remark

WLOG, all formulas have no negations ($\neg$) and no atoms of the form ($f_i \neq 0$). From now on, all the considered formulas will satisfy this.

Definition

- A *purely conjunctive* formula is a formula without disjunctions. It can be written as $\bigwedge_{i \in I}(f_{a_i} \sim_i 0)$ with $\sim_i \in \{<, \leq, =, \geq, >\}$.
- Semialgebraic sets described by a purely conjunctive formula are *basic*.
- A *lax* formula is a formula where all atoms are of the form ($f_i \leq 0$), ($f_i = 0$) and ($f_i \geq 0$).
- The *size* of a formula is the number of atoms in it.

Observation

Every closed semialgebraic set can be described by a lax formula.

Why do we care about semialgebraic sets?

1. Semialgebraic sets are a large class of sets preserved under many of the usual operations that one can do with sets (intersection, union, complementation, projection,...).
2. Semialgebraic sets can be used to describe:
   - Configuration space of a robotic arm.
   - Realization space of a polytope.
   - Configuration space of a molecule.
   - Regions of behavior of a real algebraic object.
   - ·
   - ·
   - ·

The Problem

Statement of the problem

Question

Given a semialgebraic set, how does its look like? Which is its shape?

Formalization: Information about shape in the homology groups H_i.

Problem

Given polynomials $p_1, \ldots, p_q$ and a semialgebraic formula Φ over $p := (p_1, \ldots, p_q)$, compute the homology groups of the semialgebraic set $W(p, \Phi)$.

Which complexity do we expect?

Input:

- Polynomial q-tuple $p := (p_1, \ldots, p_q)$, with p_i of degree d_i
- Formula Φ over p

Output:Homology groups of $W(p, \Phi)$

Important parameters:

- $n :=$ number of variables
- $q :=$ number of distinct polynomials in the formula
- $D := \max\{d_i \mid 1 \leq i \leq q\}$
- $N := \sum_{i=1}^{q} \binom{n+d_i}{n} = q\mathcal{O}(D)^n$ (Dim. space of tuples of polynomials)

About the measure of time:
We measure time in number of algebraic operations. We don't consider time in terms of bit size for now.

What is known?

Theorem (Gabrielov-Vorobjov)

The sum of the Betti numbers (rank of the free part of the homology groups) of $W(p, \Phi)$ *is bounded by* $\mathcal{O}(q^2 D)^n$.

Observation

The above bound is sharp. It cannot be improved in general.

Observation

We want algorithms with running time exponential in n and polynomial in q and D. Equivalently, we want polynomial time in N and exponential in n.

What is known?

However, up to now, we only have the following:

1. *Cylindrical Algebraic Decomposition* computes the homology groups of any semialgebraic set in $(qD)^{2^{\mathcal{O}(n)}}$ time. (Collins, 1975) (Wüthrich, 1976)
2. The Euler characteristic (in the lax case) can be computed in $(nqD)^{\mathcal{O}(n)}$ time. (Basu, 1996)
3. The first $\ell + 1$ Betti numbers of a semialgebraic set can be computed in $(qD)^{n^{\mathcal{O}(\ell)}}$ time. (Basu, 2006)

Except 2, all these algorithms are doubly exponential!

Observation

All the above algorithms are symbolic.

Numerical approach

Symbolic algorithms

Exact algebraic and symbolic manipulations of the data.

Advantages:

- Uniform complexity, it only depends on size of input.
- Algorithm works in all inputs.

Disadvantages:

- Symbolic small complexity does not translate always into bit small complexity.

Advantage/Disadvantage:

- They come together with constructive/structural theorems.

Numerical algorithms

Approximate algebraic and symbolic manipulations of the data.

Advantages:

- Usually, robust against errors in the input. (*Stable*).
- Stability results lead to numerical small complexity converted to bit small complexity.

Disadvantages:

- Non-uniform complexity, it depends on a property of the input that varies with the input, called *condition*.
- There are *ill-posed* inputs for which the algorithm does not work.

Advantage/Disadvantage:

- They come together with existential/structural theorems.

Condition (in a discrete setting)

$\mathcal{C}(\mathrm{input})$ controls the complexity of the algorithm:

1. Large condition $\Rightarrow$ Big time and large precision needed.
2. Small condition $\Rightarrow$ Little time and small precision needed.

In many cases, $\mathcal{C}(\mathrm{input})$ measures how bad the input is.

Condition Number Theorem:

- $\Sigma := \{\mathrm{input} \mid \mathcal{C}(\mathrm{input}) = \infty\}$. (Set of ill-posed inputs).
- $1/\mathcal{C}(\mathrm{input}) = d(\mathrm{input}, \Sigma)$ for some appropiate metric d.

Note: Condition in a continuous setting is different, but many times satisfies the above.

How do we get rid of the condition?

We endow the space of inputs with a ”natural”/”useful” distribution and study the random variable

$$\mathcal{C}(\mathrm{rinput})$$

and the corresponding bound for time $T(\mathcal{C}(\mathrm{rinput}))$.

Several notions of probabilistic complexity:

- *Average complexity*. $\mathbb{E}(T(\mathcal{C}(\mathrm{rinput})))$.
- *Weak complexity*. $\mathbb{E}(T(\mathcal{C}(\mathrm{rinput})) \mid \mathrm{rinput} \notin E_{\mathrm{size}(\mathrm{rinput})})$ where E_k is a set with probability decaying exponentially fast (*black swans*). (Ameluxen, Lotz; 2015)

Numerical progress until now

The progress until now:

1. The homology groups of a real projective variety can be computed numerically in weak exponential time. (Cucker, Krick, Shub; 2017)
2. The homology groups of a basic semialgebraic set can be computed numerically in weak exponential time. (Bürgisser, Cucker, Lairez; 2017)

Solution for the lax case

The algorithm

Main steps:

1. Sample clouds of points $\mathcal{X}_i^{\propto}$ approximating $W(p, (p_i \propto 0))$.
   Smale-Niyagi-Weinberg Sampling theory, lower bounds on reach by condition, containment relations between algebraic and metric neighborhoods
2. Convert the clouds of points $\mathcal{X}_i^{\propto}$ into simplicial complexes $\mathfrak{C}_i^{\propto}$ homologically equivalent to the $W(p, (p_i \propto 0))$.
   Nerve theorem, Čech complex
3. Construct $\Phi(\mathfrak{C}_i^{\propto} \mid i \in [q], \propto \in \{\leq, =, \geq\})$ homologically equivalent to $W(p, \Phi)$.
   Functiorality, quantitative Durfee's theorem, Thom-Whitney theory
4. Compute homology.

Hidden technical steps:

1. Reduction to homogeneous case in the sphere.
2. Estimation of the condition.

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Condition used

The condition of our input, $\overline{\kappa}_{\text{aff}}$, only depends on the tuple of polynomials p not on the formula.

Well-posed inputs (i.e. those with $\overline{\kappa}_{\text{aff}}(p) < \infty$):

For every $L \subseteq [q]$, the projective real zero set $\mathrm{V}_{\mathbb{P}}(p^L)$ is regular and transversal to the hyperplane at infinity.

Example

p quadratic polynomial:

1. $p^{-1}(0)$ hyperbola $\Rightarrow$ p well-posed ($\overline{\kappa}_{\text{aff}}(p) < \infty$)
2. $p^{-1}(0)$ parabola $\Rightarrow$ p ill-posed ($\overline{\kappa}_{\text{aff}}(p) = \infty$)
3. $p^{-1}(0)$ ellipse $\Rightarrow$ p well-posed ($\overline{\kappa}_{\text{aff}}(p) < \infty$)

Distribution considered

$\mathcal{P}_{\boldsymbol{d}}[q] :=$ q-tuples of polinomials p with p_i of degree $\leq d_i$
$\mathcal{H}_{\boldsymbol{d}}[q] :=$ q-tuples of hom. polinomials p with p_i of degree d_i

Bombieri-Weyl norm:An inner product norm on $\mathcal{H}_{\boldsymbol{d}}[q]$ that is invariant under orthogonal transformations, i.e., for every $g \in O(n+1)$ and $p \in \mathcal{H}_{\boldsymbol{d}}[q]$, $\|p \circ g\| = \|p\|$. Restricts to $\mathcal{P}_{\boldsymbol{d}}[q]$ via homogenization.

We consider the uniform distribution on $\mathbb{S}(\mathcal{P}_{\boldsymbol{d}}[q]) = \mathbb{S}^{N-1}$ where sphere is taken with respect the Bombieri-Weyl norm.

Why? Because it does not favor any direction in space.

Issue: One can consider other distributions for other valid reasons.

1. Each polynomial in the tuple sampled independently.
2. Coefficients uniformly taken from $[-1, 1]$.
3. Coefficients uniformly taken from $[-M, M] \cap \mathbb{Z}$.

Main result

Theorem

There is a numerical algorithm Homology, *numerically stable, that, given a tuple* $p \in \mathcal{P}_{\mathbf{d}}[q]$ *and a lax Boolean formula* Φ *over* p*, computes the homology groups of* $W(p,\Phi)$*.The cost of* Homology *on input* (p,Φ) *denoted* $\mathrm{cost}(p,\Phi)$*, that is, the number of arithmetic operations and comparisons in* $\mathbb{R}$*, satisfies:*

(i) $\mathrm{cost}(p,\Phi) \leq \mathrm{size}(\Phi) q^{\mathcal{O}(n)} (nD\overline{\kappa}_{\mathrm{aff}}(p))^{\mathcal{O}(n^2)}$.

Furthermore, if p *is drawn from the uniform distribution on* $\mathbb{S}^{N-1}$*, then:*

(ii) $\mathrm{cost}(p,\Phi) \leq \mathrm{size}(\Phi) q^{\mathcal{O}(n)} (nD)^{\mathcal{O}(n^3)}$ *with probability at least* $1-(nqD)^{-n}$*, and*

(iii) $\mathrm{cost}(p,\Phi) \leq 2^{\mathcal{O}\left(\mathrm{size}(p,\Phi)^{1+\frac{2}{D}}\right)}$ *with probability at least* $1-2^{-\mathrm{size}(p,\Phi)}$.

What's left to do?

1. The non-lax case.
2. Other probability distributions on the input space.

Bere arretagatik eskerrik asko!
¡Gracias por su atención!
Thank you for your attention!
Vielen Dank für Ihre Aufmerksamkeit!

Galderak?
¿Preguntas?
Questions?
Fragen?